Algorithmic of translation surfaces

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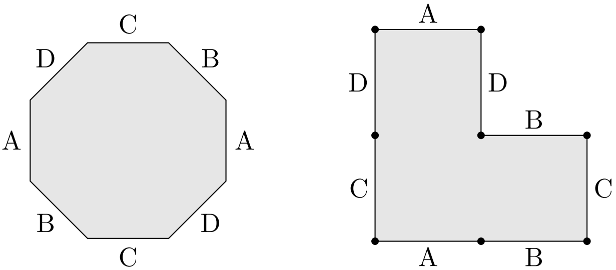
What and why

What is a translation surface?

definition: edge to edge gluings by translation.



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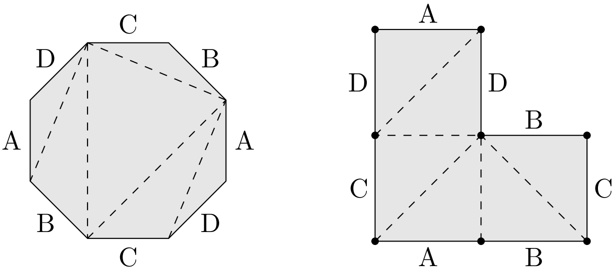


convenient encoding: T a triangulation, $E(T)$ = oriented edges. A *translation structure* is $e \in E(T) \mapsto v(e) \in \mathbb{R}^2$ such that

- for each edge e : $v(-e) = -v(e)$,
- for each triangle e_1, e_2, e_3 : $v(e_1) + v(e_2) = -v(e_3)$ and $\det(e_1, e_2) > 0$.

Why do we study translation surfaces?

- simpler than hyperbolic geometry
- same flavour than embedded graphs geometry
- (algebraic geometry) a translation surface is a Riemann surface endowed with a nonzero holomorphic one form
- (dynamics) polygonal billiards

Goal of the talk

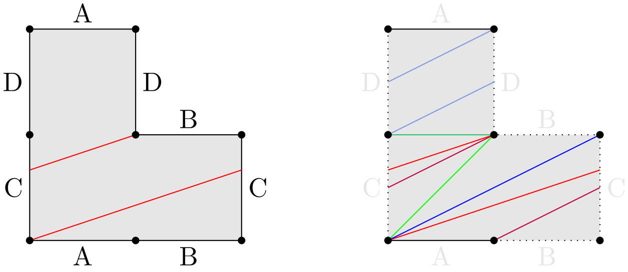
- algorithmic of translation surfaces and implementation in `https://flatsurf.github.io/`
- four open questions and two conjectures

Geometry

Saddle connections and flat triangulations

Definition

saddle connection : straight line segment joining two conical singularities.
flat triangulation : triangulation of the translation surface whose edges are saddle connections



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Theorem (Masur)

Any two flat triangulations of M can be joined by a sequence of edge flips.

Delaunay triangulation

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conjecture 1: Given $(T, \{v(e)\}_{e \in E(T)})$, one can compute the Delaunay triangulation in $O(\log \max_{e \in E(T)} \|v(e)\|)$.

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conjecture 2: There is a $O(L^2)$-algorithm (aka optimal) based on flips in triangulation.

Tightening geodesics

Theorem (folklore)

A curve in a translation surface is a geodesic if and only if it is a concatenation of straight line segments meeting at conical singularities with angles $\geq \pi$*.*
Given a path which is a concatenation of saddle connections, there is a tightening procedure to homotope the path to a geodesic.

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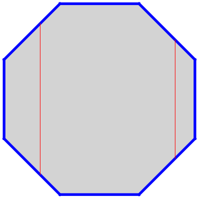
Consequence: one can compute distances in the universal cover of M and in particular the bottom of the length spectrum (not yet in `flatsurf`).
open questions 1 and 2:

- complexity of tightening (possible combinatorial explosion)
- how hard it is to approximate the volume entropy?

Translation flow ($\mathbb{R}$-action on M)

Translation flow



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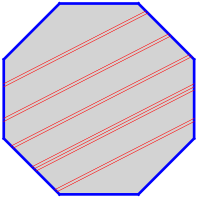

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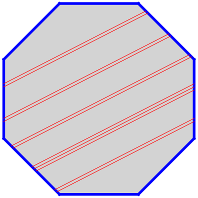

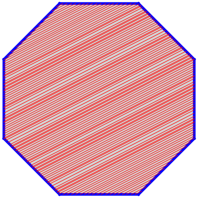

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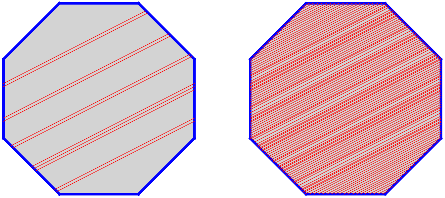

Theorem (Katok, Keane)

M a translation surface. $\phi_M^t : M \to M$ its translation flow. $x \in M$ with infinite orbit. Then $\overline{\{\phi_M^t(x) : t \geq 0\}}$ is either

- *a* **circle** *(iff x is a periodic point)*
- *or* **a subsurface bounded by finitely many saddle connections**.

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consequence: the surface decomposes into finitely many

- cylinders,
- minimal components
- saddle connections.

A partial result

Theorem (VD + Julian Rüth)

There exists a semi-algorithm that compute and certify the decomposition of the surface into cylinders + minimal components + saddle connections.

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$GL_2(\mathbb{R})$-action on the moduli space of translation surfaces

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partial result : there exists a semi-algorithm to compute $\overline{GL_2(\mathbb{R}) \cdot M}$ (available in `flatsurf`).