Energy distribution of harmonic 1-forms on thick & thin parts

Luminy May 4, 2022

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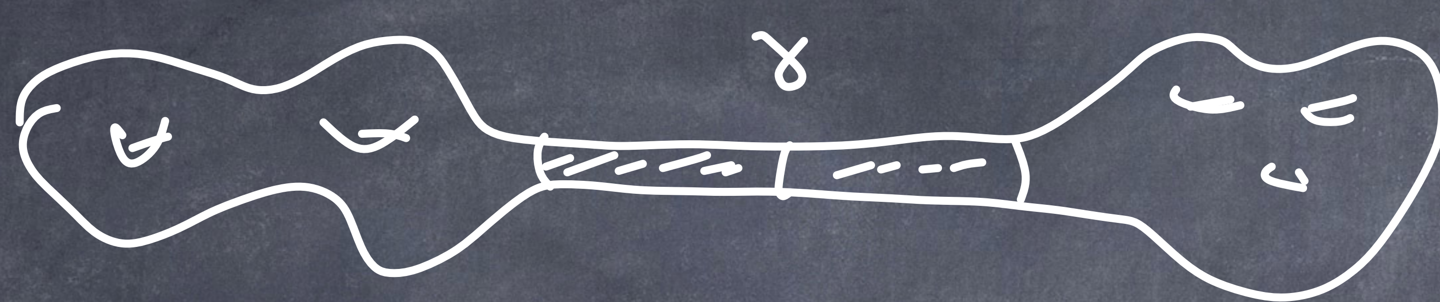
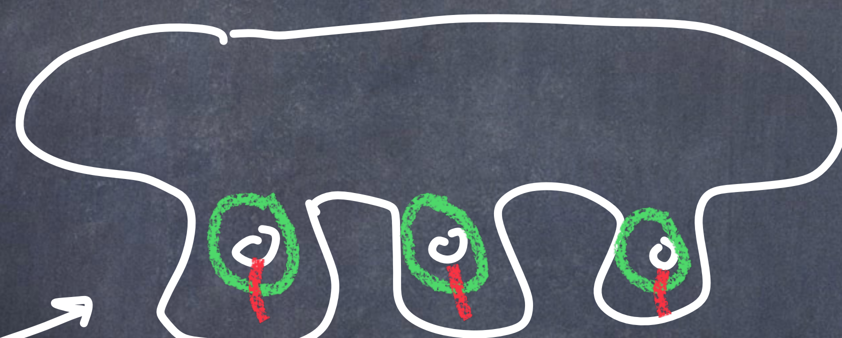
(Math Z, 2021)

Thin part



① Jacobian & Period matrix



canonical homology bas

$$i(\alpha_{2k-1}, \alpha_{2k}) = 1$$

$$i(\alpha_{2k}, \alpha_{2k-1}) = -1$$

Intersection matrix

$$I_g = \begin{bmatrix} \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & & 0 \\ & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \end{bmatrix}$$

Take dual basis of harmonic forms

$\sigma_1, \dots, \sigma_{2g}$ $\qquad \int_{\alpha_i} \sigma_j = \delta_i$

$P_S = (p_{ij})$, $\qquad p_{ij} = \int_S \sigma_i \wedge * \sigma_j$

P_S is symmetric & positive definite

$\hookrightarrow$ Flat Riemannian metric on 2g-torus

$\mathbb{R}^{2g}$

$g_{ij} = p_{ij}$ on $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$

Stokes theorem yields:

$$P_s I_g P_s = I_g \qquad (P_s \text{ symplectic})$$

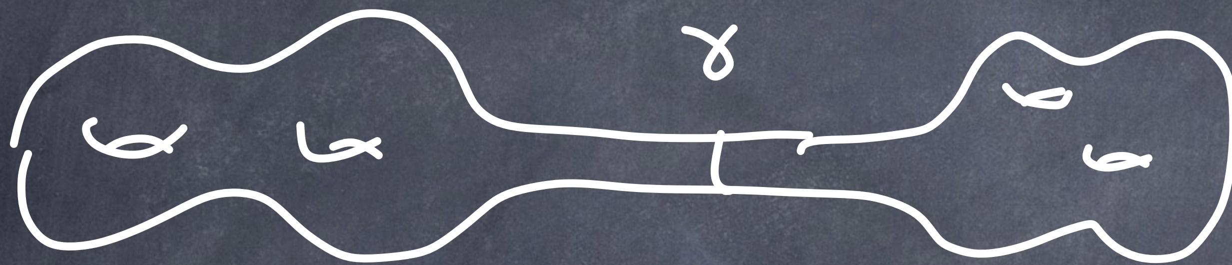
Famous unsolved Schottky problem: can you characterize those symplectic P that arise from a R.S. ?

Markus 1963:

S_1 S_2

$$\begin{bmatrix} P_{S_1} & 0 \\ 0 & P_{S_2} \end{bmatrix}$$ not the PM of a R.S.

② Thin separating part

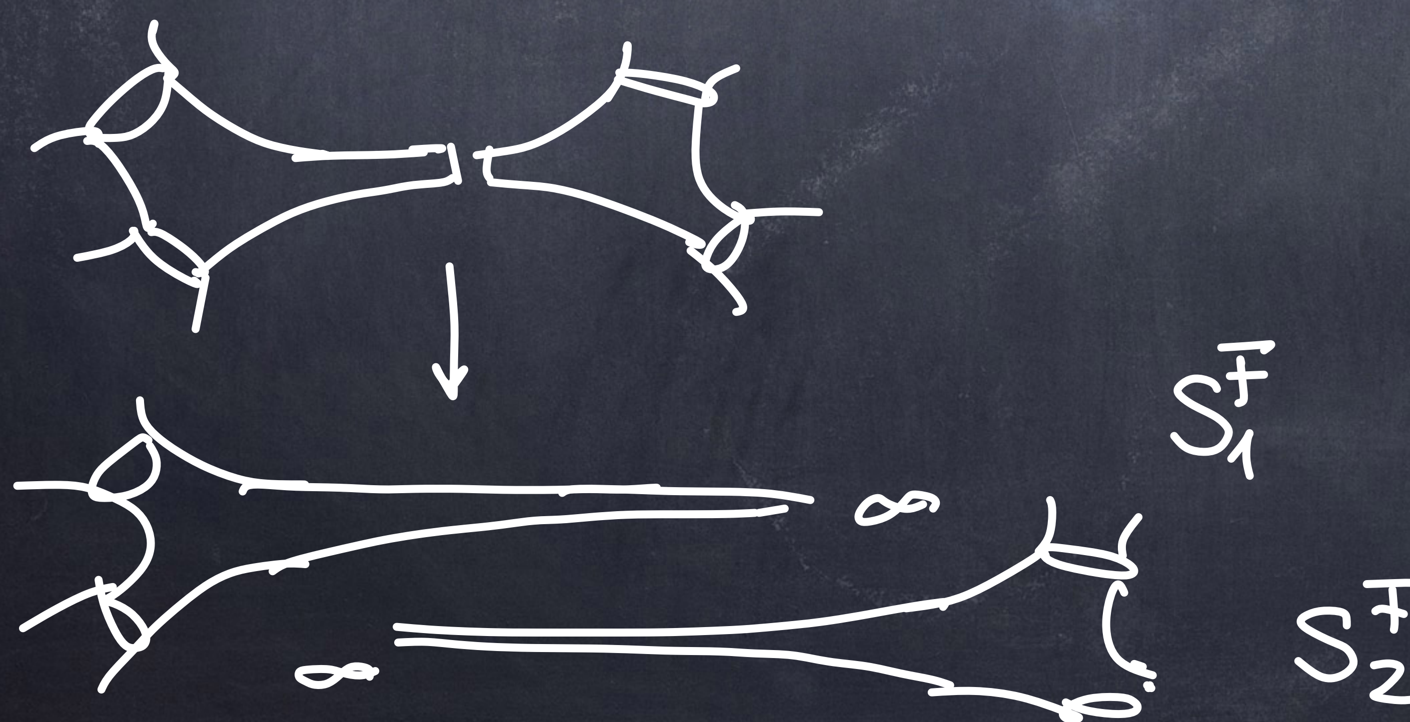


$\ell(\gamma)$ small

how close is P_s to $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$?

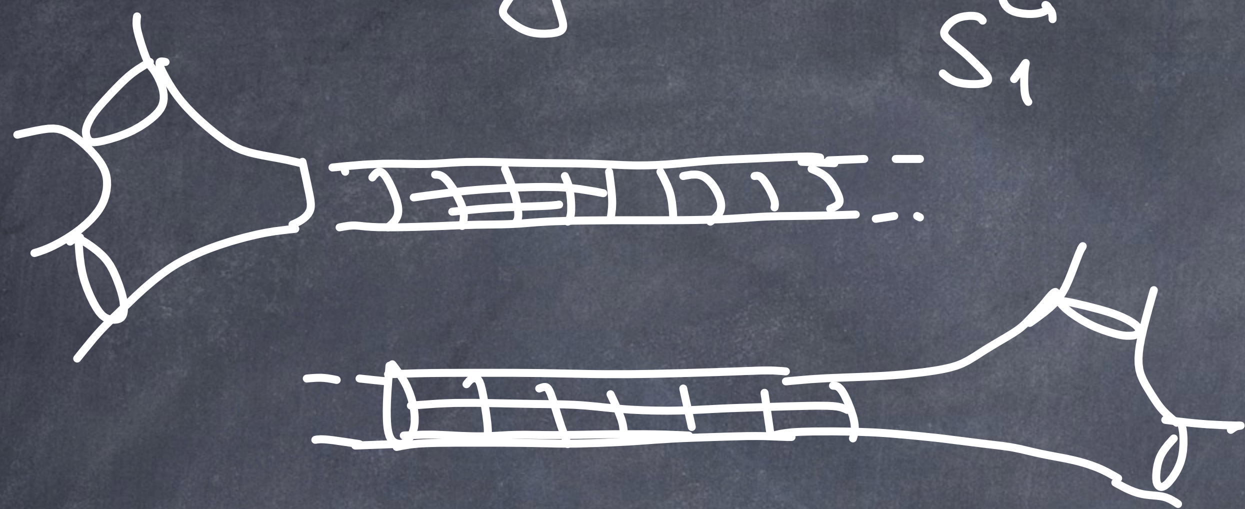
Go to the boundary of $\mathcal{M}_g$

F) Fenchel-Nielsen limit:



Colbois-Courtois
Laplace spectrum
1989

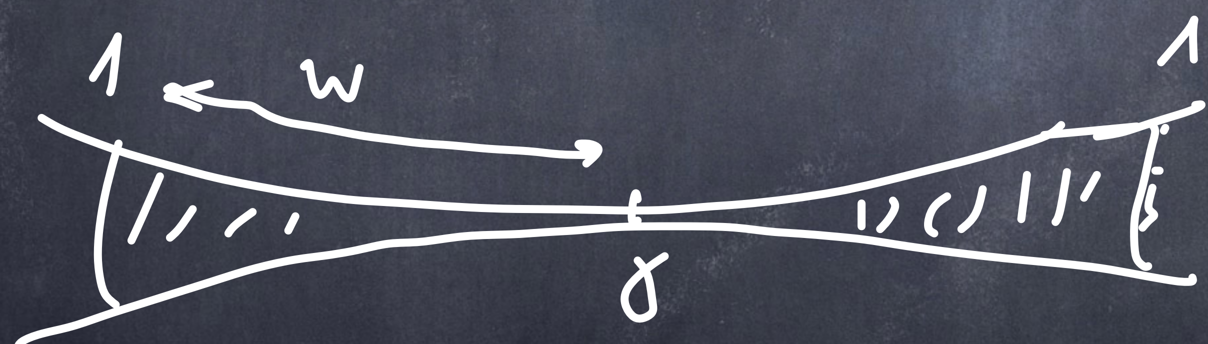
G) Grafting limit



S_1^G

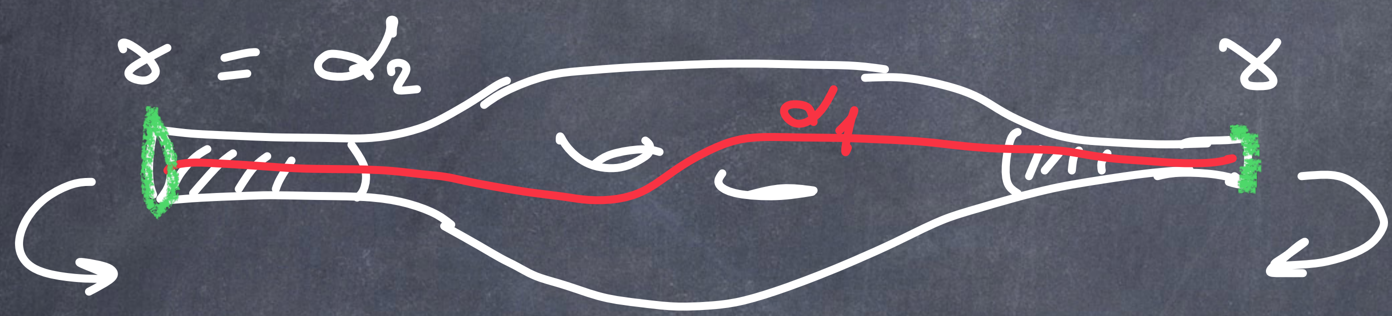
S_2^G

<u>Thm</u> $$\mathcal{P}_S = \begin{bmatrix} \mathcal{P}_{S_1^G} & 0 \\ 0 & \mathcal{P}_{S_2} \end{bmatrix} + \mathcal{O}\left(e^{-2\pi^2/\ell(\gamma)}\right)$$



$$\ell(\gamma) = e^{-w} \qquad \mathcal{O} = \mathcal{O}\left(e^{-2\pi^2 e^{w}}\right)$$

③ Non-separating case

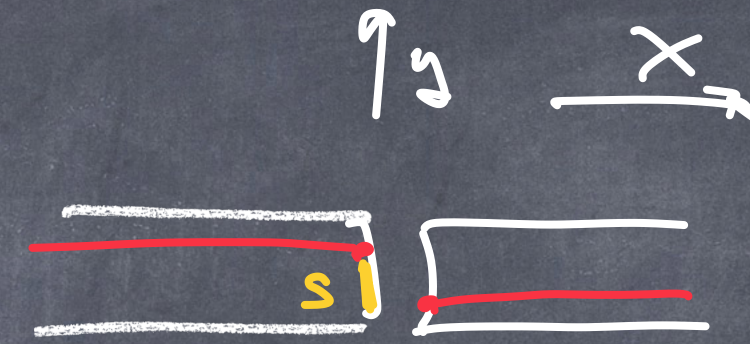


Here comes trouble:

- S^G has smaller genus
- $P_{22} = \int_S \sigma_2 \wedge * \sigma_2 \to \infty$
- $\int_{\alpha_1} \sigma_2$ very sensitive to twist

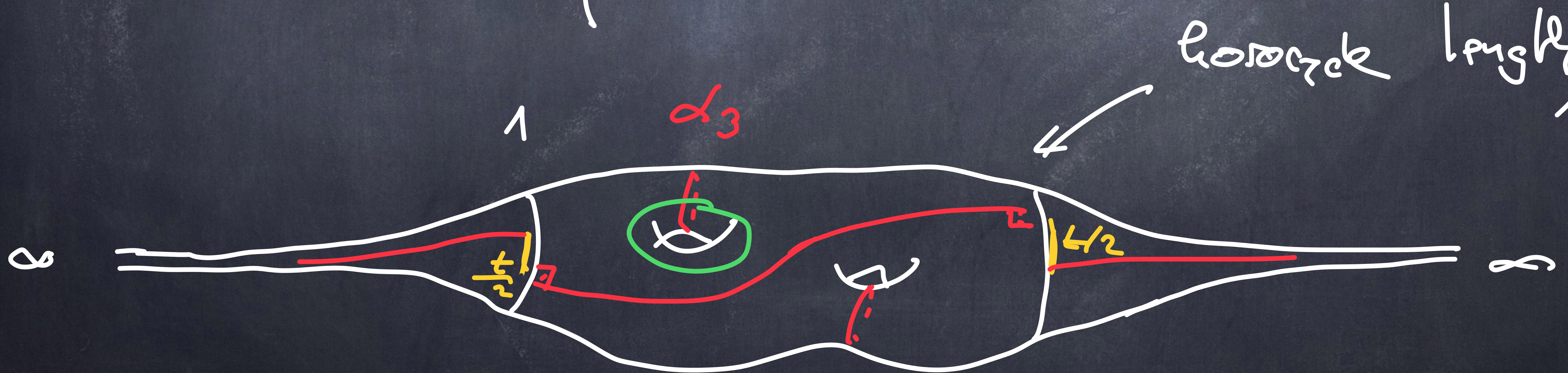
Namely:

over night



$$\int_{\alpha_1} \sigma_2 \quad \text{changes by} \quad \int_s dy$$

④ Twist at infinity of R with two cusps



t = twist at ∞.

⑤ Blown up period matrix for $\mathcal{R}$

Forms $\tau_3, \tau_4, \ldots, \tau_{2g-1}, \tau_{2g}$ as usual

τ_1 the exact harmonic form with logarithmic poles,

$$\tau_1 = dx + \text{exponential decay}$$

$$*\tau_1 = dy + \text{exponential decay}$$

$$\tau_2 = *\tau_1 - \sum_{k=3}^{2g} \left(\int_{\alpha_k} *\tau_1 \right) \tau_k$$

Great fact:

$$q_{ij} := \int_R \tau_i \wedge * \tau_j \quad \text{finite} \quad i,j = 2,\dots,2g$$

Set also $q_{1j} = q_{j1} = 0$

except $q_{22} = \infty$

but

$$\pi_{22} := \int_R (\tau_2 - *\tau_1) \wedge * (\tau_2 - * \tau_1)$$

finite.

Set

$$\mathcal{K}_j = -\int_{A_1} \tau_j \qquad j = 2,\dots,2g$$

$$\mathcal{K}_1 = 1$$

and define function of variable λ:

$$P_{ij}(\lambda) = q_{ij} + \frac{K_i K_j}{\lambda}$$

$$P_{22}(\lambda) = \pi_{22} + \lambda + \frac{K_{22} K_{22}}{\lambda}$$

$$\mathcal{P}_{\mathcal{R}}(\lambda) = \left(P_{ij}(\lambda) \right) \quad \text{a } 2g \times 2g \text{ matrix}$$

6 **Theorem**

$$\mathcal{P}_S = \mathcal{P}_{SG}(m + L_\gamma) + \mathcal{O}(e^{-\pi L_\gamma})$$

7 Remark: The $P_R(\lambda)$ are symplectic

Proof

$$I_g = P_{S_L} I_g P_{S_L} = P_R^{(m+L)} + O(e^{-\pi L}) \cdot I_g \cdot (P_R(m+L) + O(e^{-\pi L}))$$

$$= P_R(m+L) \cdot I_g P_R(m+L) + O(e^{-\pi L})$$

rational function of L

exponential

End