Computing complete hyperbolic structures on cusped 3-manifolds

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May 16, 2022

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With several mistakes on the way. . .

Knots

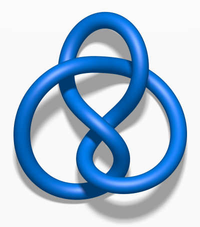

Figure-eight knot[1]

Ambient isotopy

Continuous distortion of the ambient space.

[1] https://en.wikipedia.org/wiki/Figure-eight_knot_(mathematics)

3-Manifolds and knots

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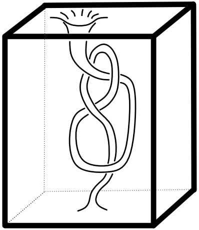

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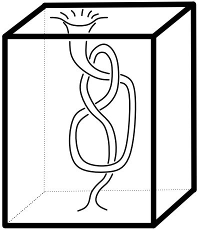

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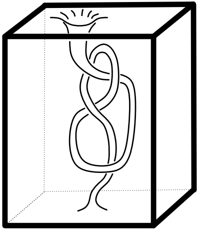

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Theorem (Thurston)

Knots are either satellites, torus or hyperbolic.

Complete hyperbolic structures

Provides a hyperbolic metric on the manifold.

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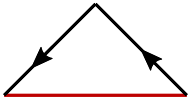
The hyperbolic volume

Great invariant linked to many theories and conjectures.

Outline

Generalized triangulations

Triangulated 3-manifolds with self-identifications.



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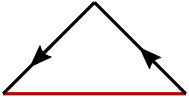
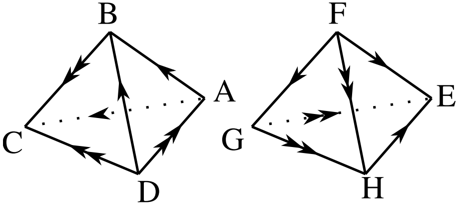
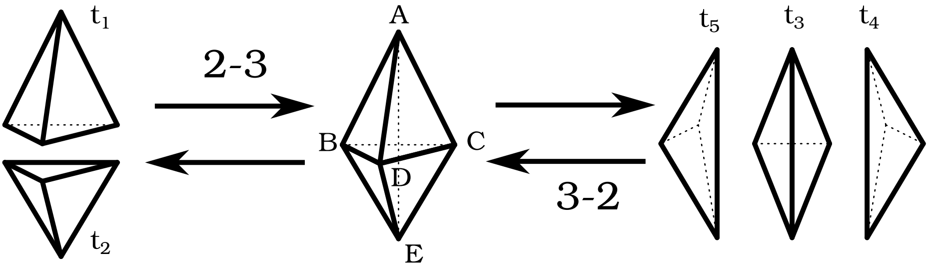


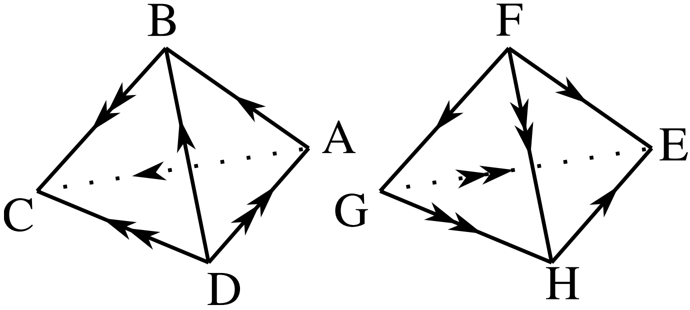


Theorem (Pachner,1991)

Any two triangulations of a piecewise linear 3-manifold can be linked by a sequence of Pachner moves.



Knot complement triangulations



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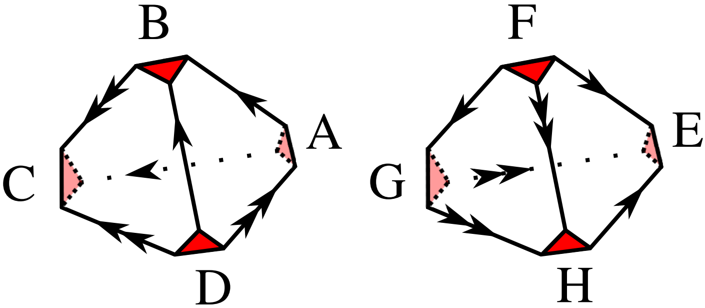

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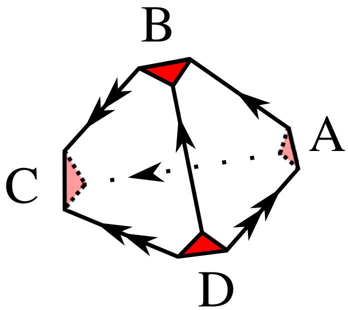
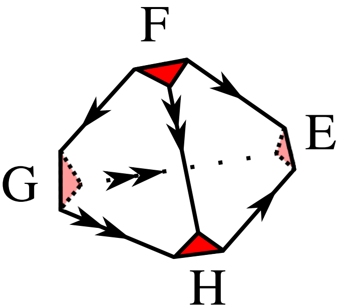
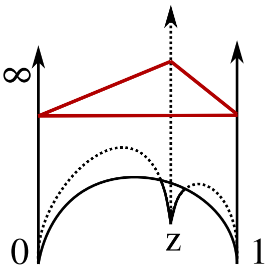




All 3-manifolds are triangulable, and there exists an algorithm for knots complements.

Hyperbolic ideal tetrahedra



Definition

A hyperbolic ideal tetrahedron is the convex hull of four distinct points on $\partial\mathbb{H}^3$.

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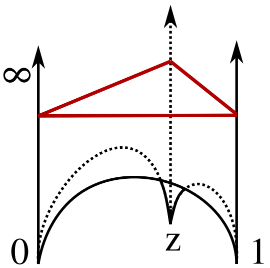

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Either described with a single complex parameter or three dihedral angles.

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Every point must have a neighborhood isometric to a sphere.

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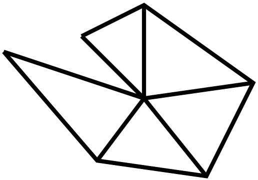

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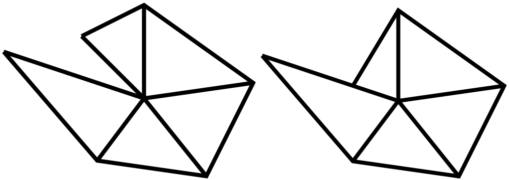

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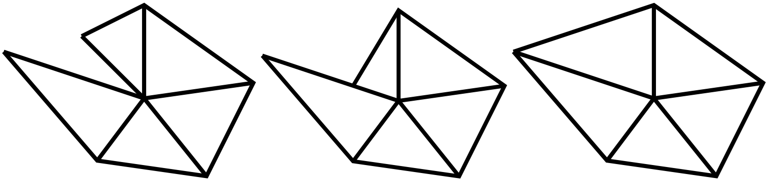
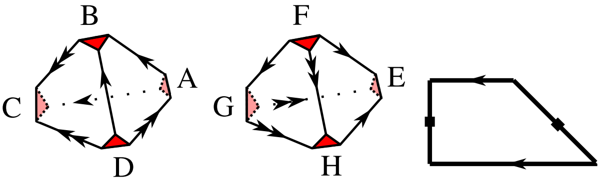


Completeness around the cusp



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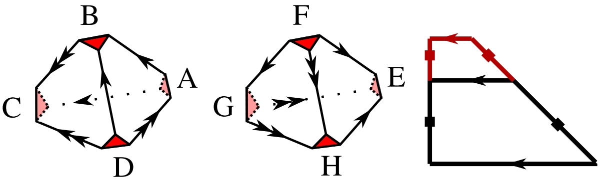

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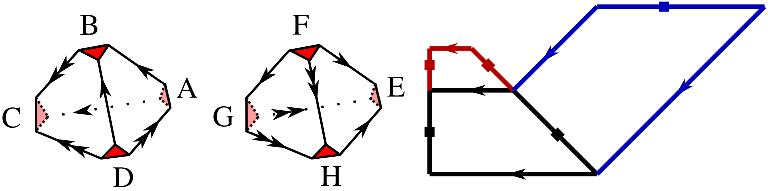

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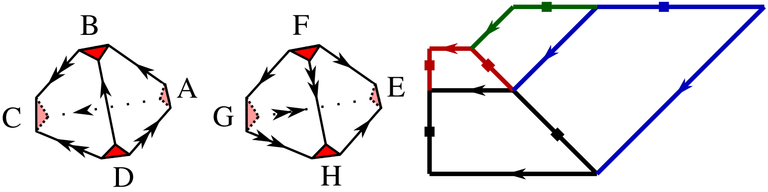

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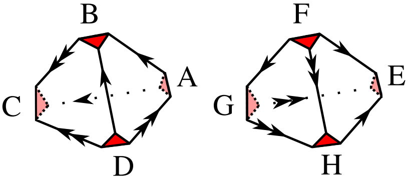


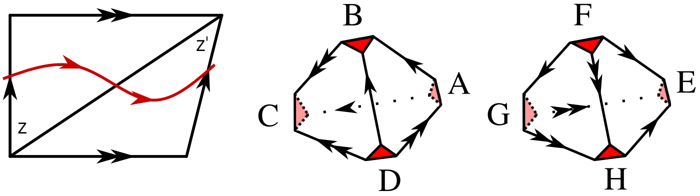
Theorem

The hyperbolic metric of the 3-manifold is complete if and only if the euclidean metric on the boundary torus is complete.

Normal curves and holonomy

Normal curve

A sequence of segments cutting the triangles only by their edges.



Thurston gluing equations (1980)

Edge equations

$$\sum_i \log(z_i) = 2i\pi$$

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Complete hyperbolic structure problem

- Input: triangulation τ of a knot complement.
- Output: complete hyperbolic structure on τ.

SnapPy

Uses Newton's method to directly solve Thurston's equations.

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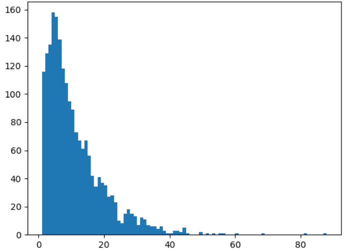
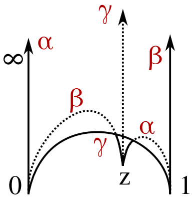
Uses Newton's method to directly solve Thurston's equations.

- No guarantees on the convergence speed.
- No studies of the failures cases.
- No methods/heuristics to find geometrizable triangulations.



The polytope of angle structures



- all angles are in $]0,\pi[$;
- the diahedral angles of the tetrahedra sum to π;
- around each edge, the angles sum to 2π.

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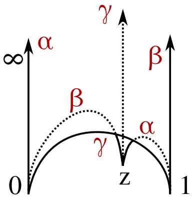

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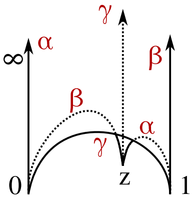

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Lemma (Neumann, 1992)

With τ the triangulation and $\mathcal{A}(\tau)$ the polytope of angle structures:

$$dim\ \mathcal{A}(\tau) = |\tau| + |\partial M|$$

Existence of a hyperbolic metric

Theorem (Casson)

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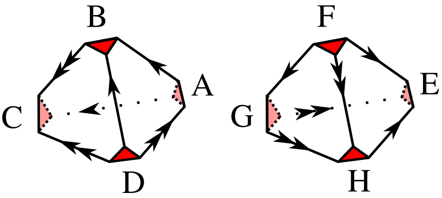
Strategy $\rightarrow$ maximize $\mathcal{V}$ over $\mathcal{A}(\tau)$.

Leading-trailing deformations



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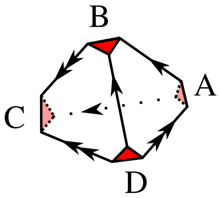
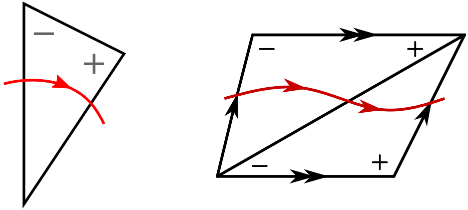
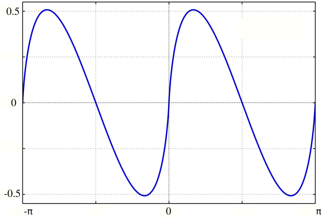




Volume of an angle structure

Lobachevsky function

$$Л(x) = -\int_0^x \log|2\sin t|\,\mathrm{d}t$$



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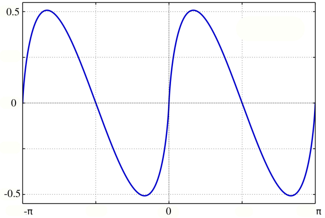

Volume of an ideal tetrahedron

$$\mathcal{V}(\alpha,\beta,\gamma) = Л(\alpha) + Л(\beta) + Л(\gamma)$$

Convexity

Optimization problem

- Maximize $\mathcal{V}$ over $\mathcal{A}(\tau)$.
- A base of $\mathcal{A}(\tau)$ is can be easily computed.
- $\mathcal{V}$ is the sum of the volumes of the tetrahedra.

Lemma (Rivin, 1994)

$\mathcal{V}$ is strictly concave on a ideal hyperbolic tetrahedron.

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- Let p_1 and p_2 the smallest angles of a tetrahedron T.
- Let w a linear transformation over the angles of T, with coefficients w_1, w_2 and w_3 such that $w_1 + w_2 + w_3 = 0$.

$$-\frac{\partial^2 \mathcal{V}(T)}{\partial w^2} = \frac{(w_1 + w_2)^2 + (w_1 \cot p_1 - w_2 \cot p_2)^2}{\cot p_1 + \cot p_2}$$

The behavior of the volume

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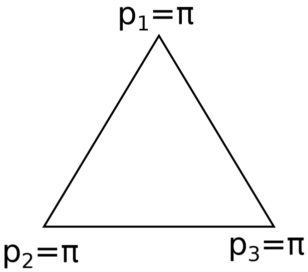

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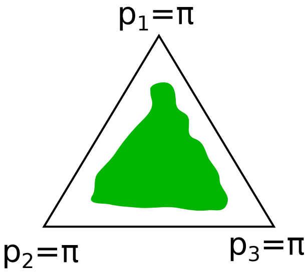

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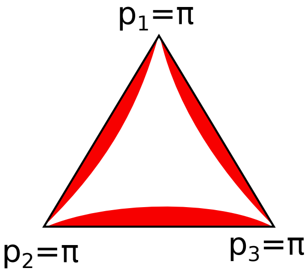
- If $\min(p_1, p_2) = x$ and $\max(p_1, p_2)$ is constant, then $\frac{\partial^2 \mathcal{V}(T)}{\partial w^2} = O_{x \to 0}(\frac{1}{x})$.



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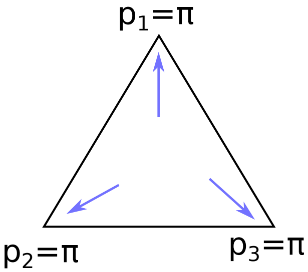
- If $(p_1, p_2) \to (0, 0)\dots$



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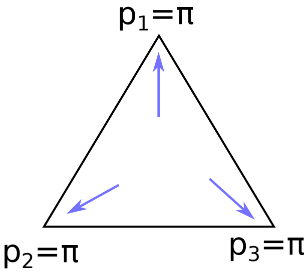
- If $(p_1, p_2) \to (0, 0)$... possible optimal on the boundary.



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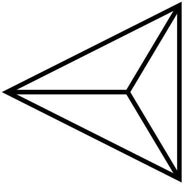

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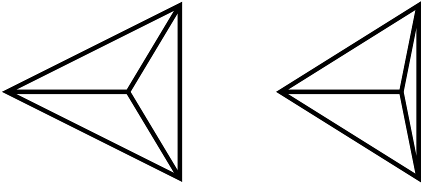

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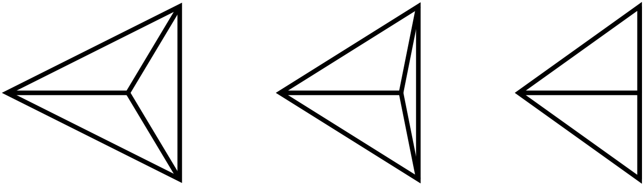

While (not geometric)
  maximize volume
  delete flat tetrahedra

Geometric Pachner moves

To try to preserve the geometric information.

Geometric Pachner move

Move between angle structures not modifying the tetrahedra not involved in the move.

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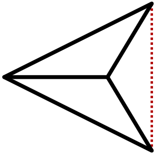
Lemma

A 3-2 move is always geometric, a 2-3 is geometric iff the "external" angles are larger than π.



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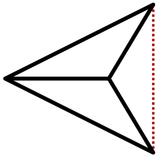

Remark

3-2 moves do not preserve the volume.

Flat tetrahedra deletion

Sequence of 2-3 moves followed by a 3-2.

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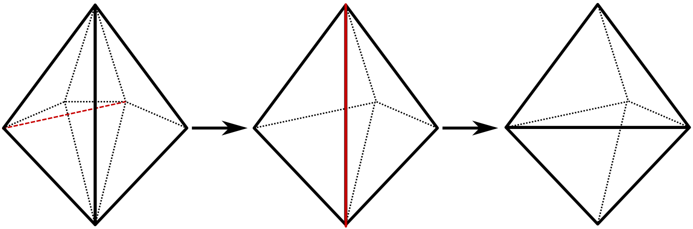
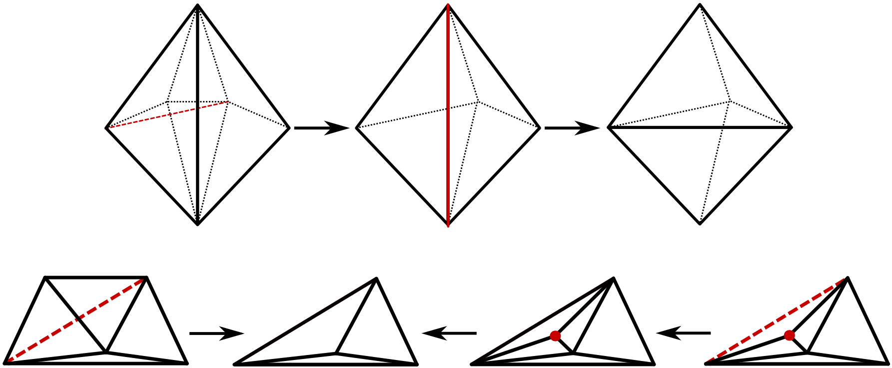
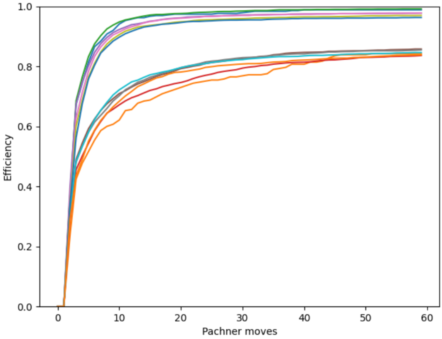


Flat tetrahedra deletion

Sequence of 2-3 moves followed by a 3-2.

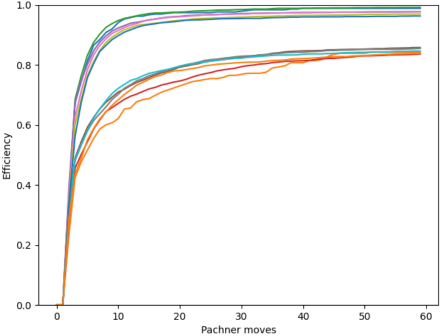


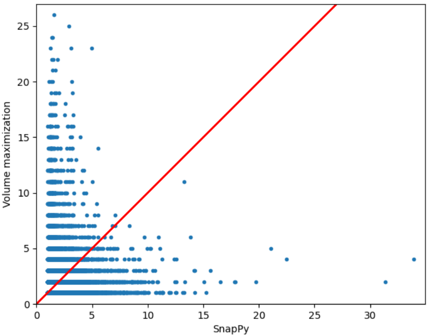
Results



Right: Number of Pachner moves required to find a complete hyperbolic structures.

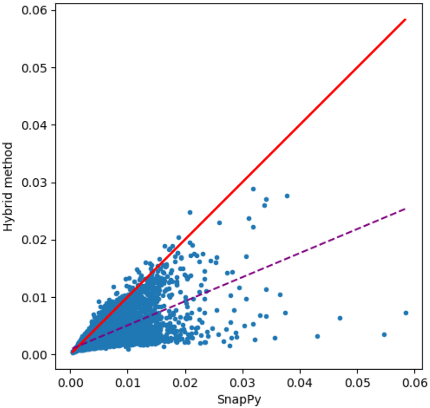
Results





Right: Number of Pachner moves required to find a complete hyperbolic structures. Left: comparison of the difficult cases with SnapPy.

Results 2



Time required to compute a complete hyperbolic structure in seconds, SnapPy compared to hybrid method.

Conclusion

Starting point

- To find a complete hyperbolic structure $\rightarrow$ solve gluing equations;
- not all triangulations admits a solution;
- there is a complete hyperbolic structure iff the hyperbolic volume is maximal.

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Method

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Results

- A lot of triangulations need few moves to accept complete hyperbolic structures;
- our method alone can be costly and not succeed;
- allows to improve on random re-triangulations when mixed.