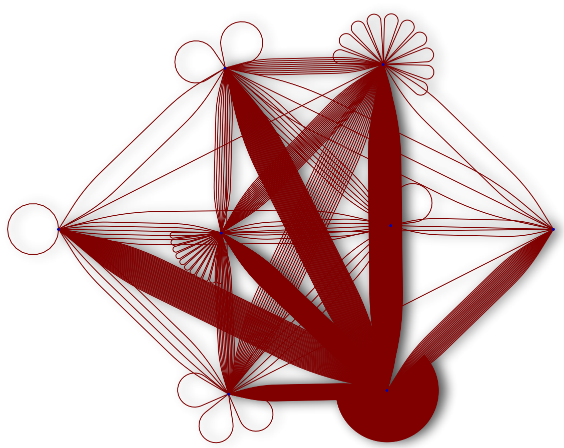
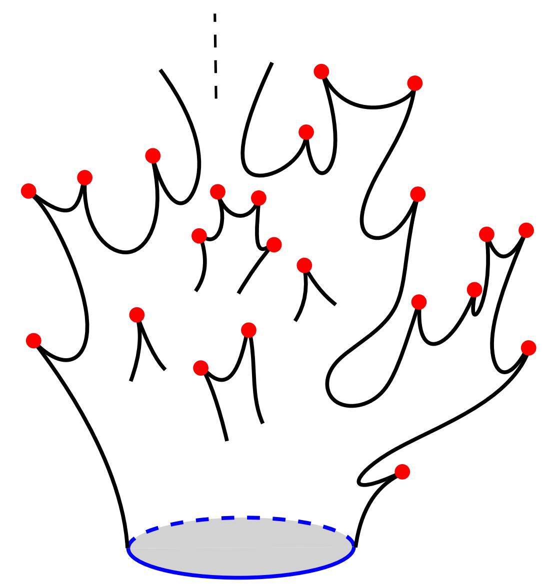
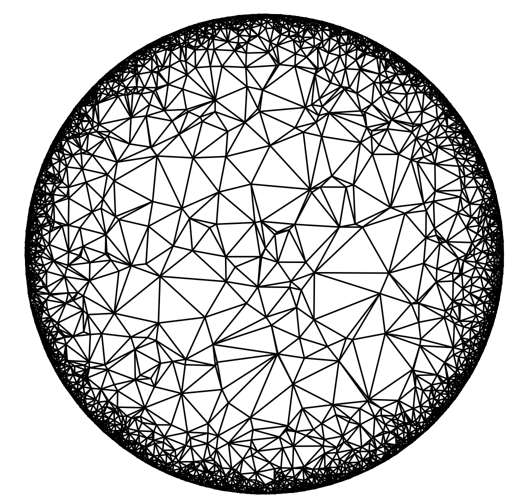
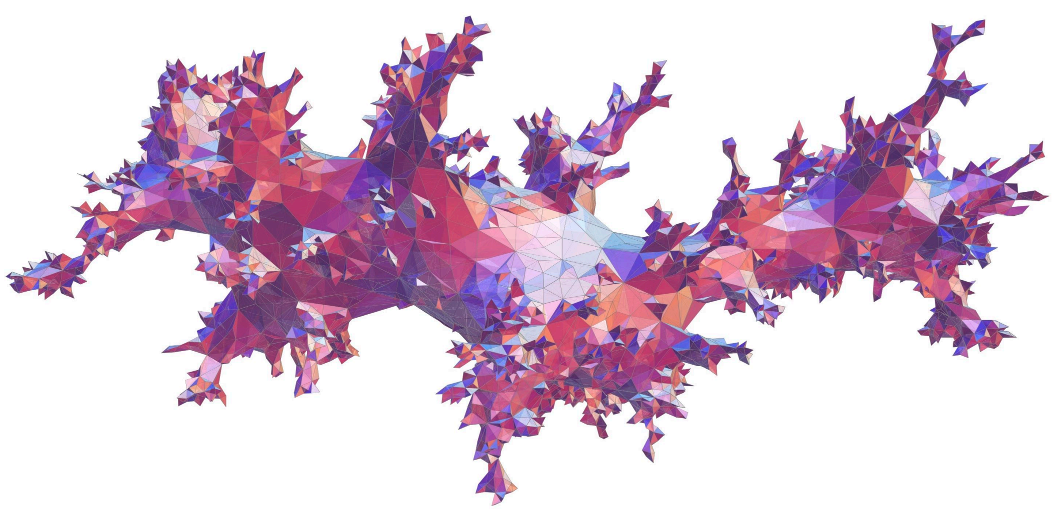
Random Maps & Hyperbolic Surfaces

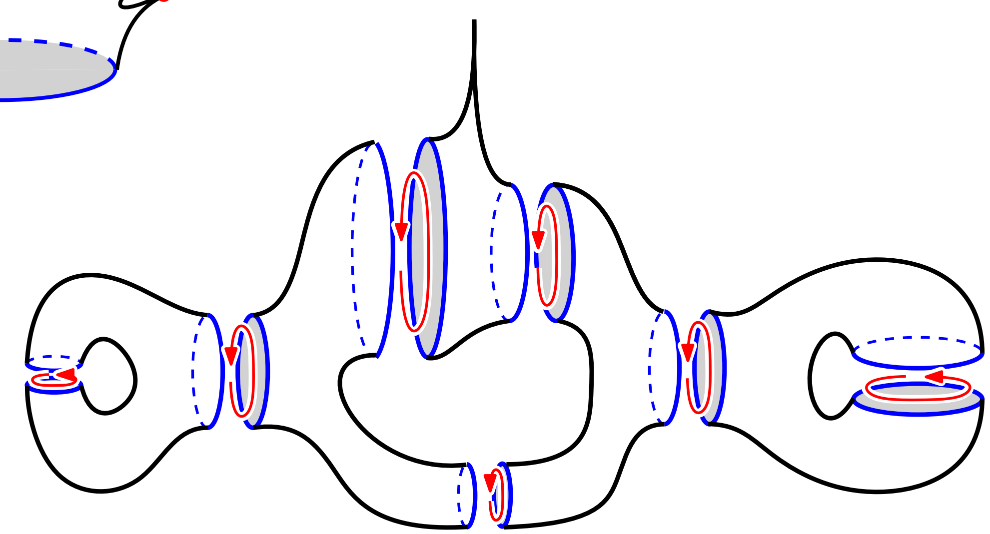
A few tableaux…









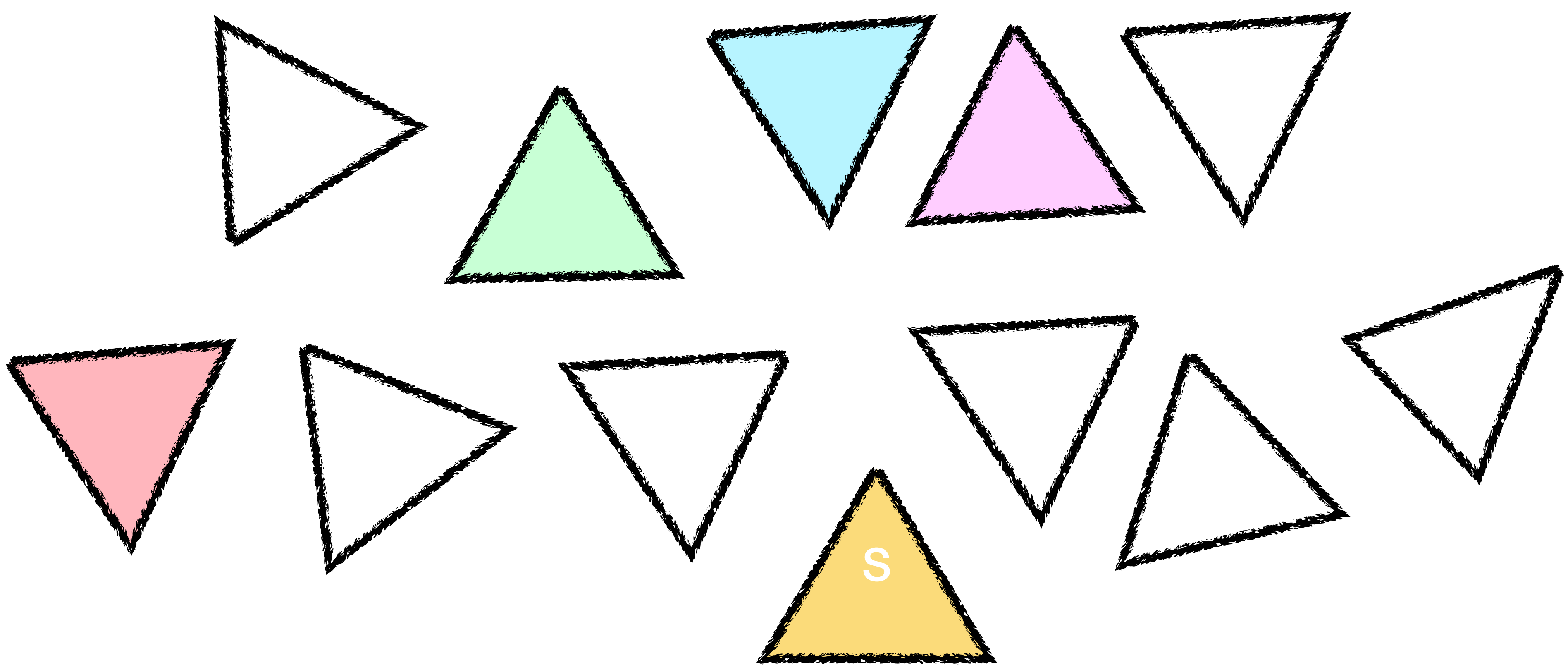


Nicolas Curien (Université Paris-Saclay)

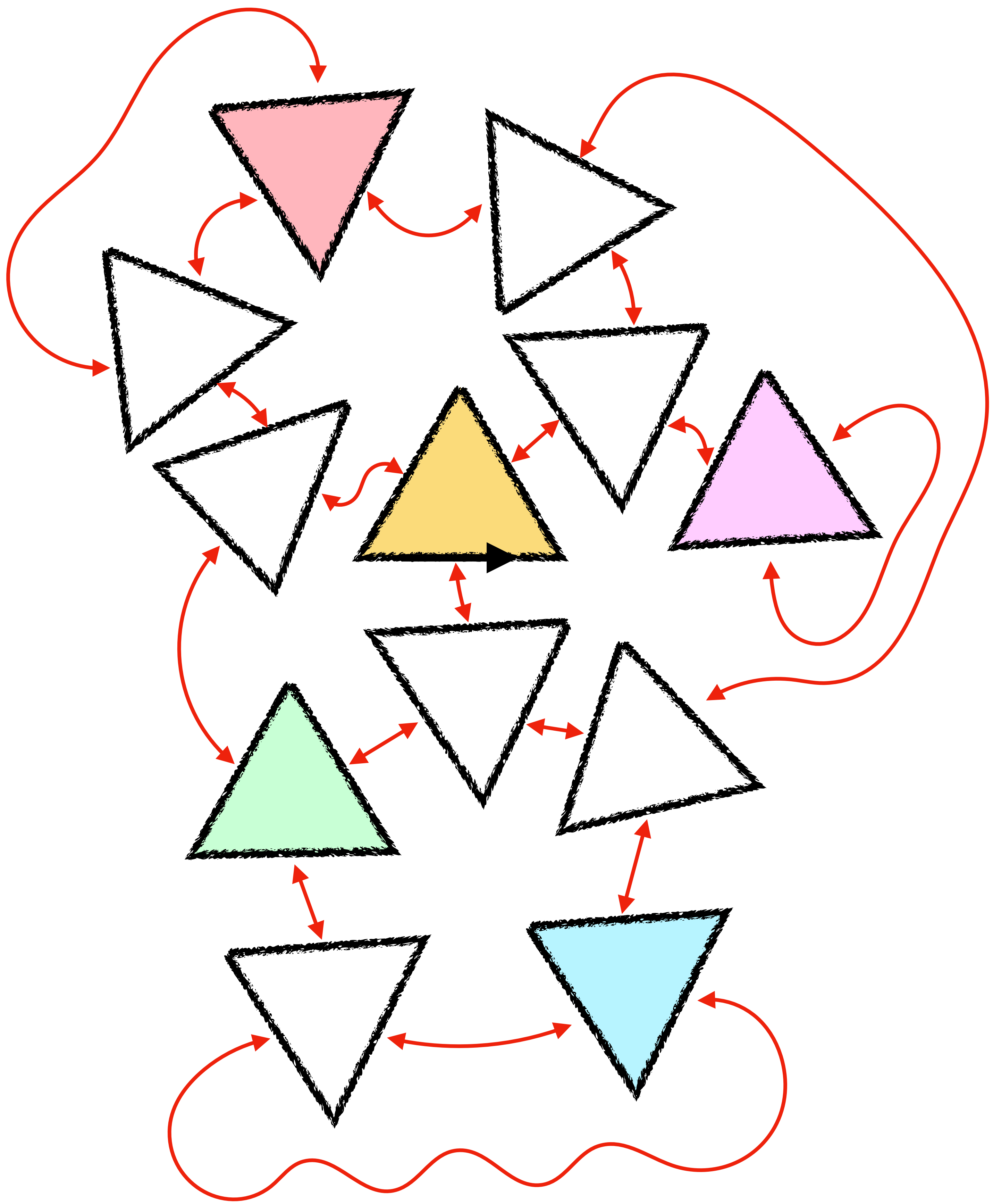
Based on joint and on-going works with T. Budd, T. Budzinski, B. Petri

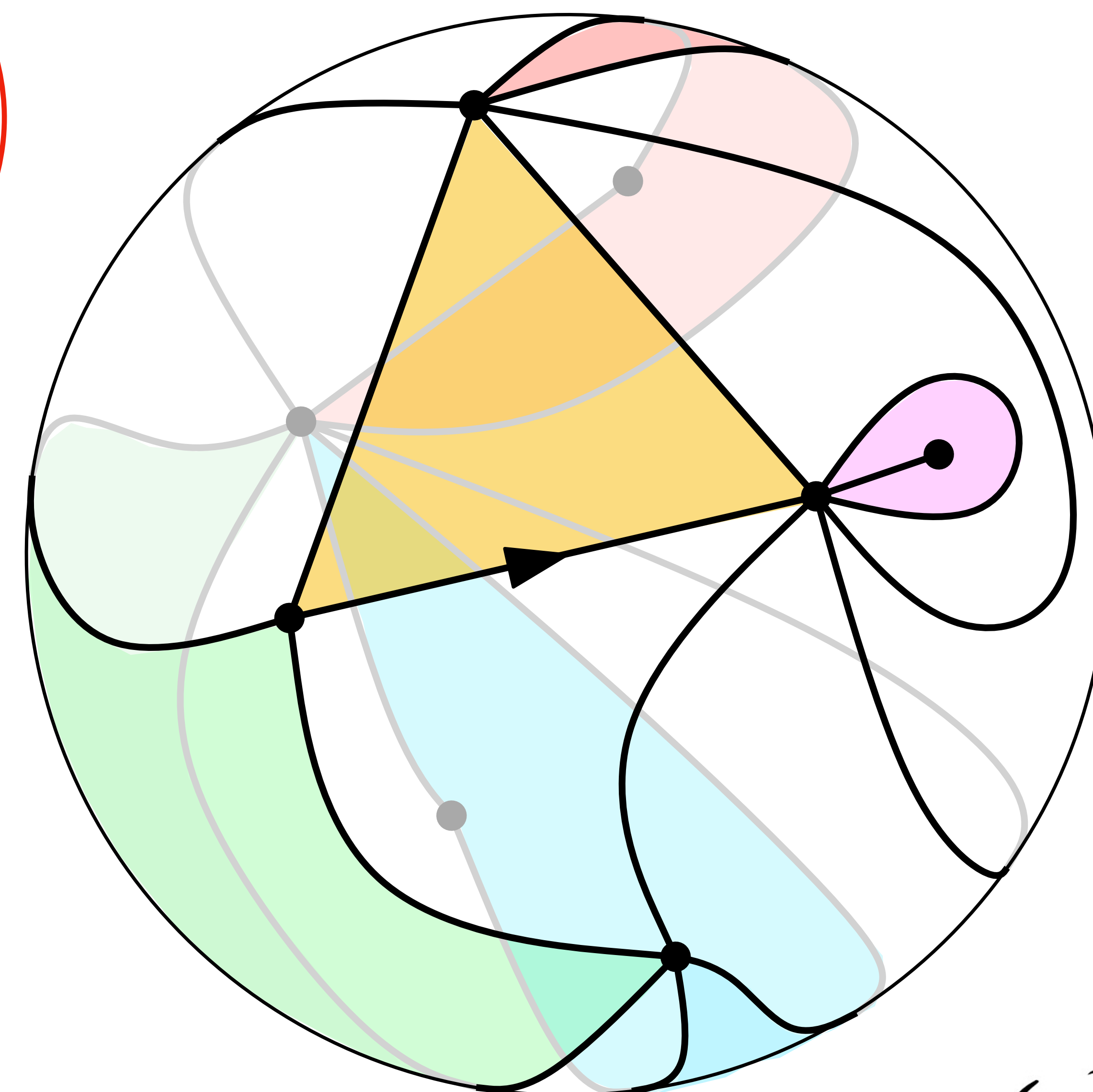
Tableau 1: Results

Part I: Random triangulations

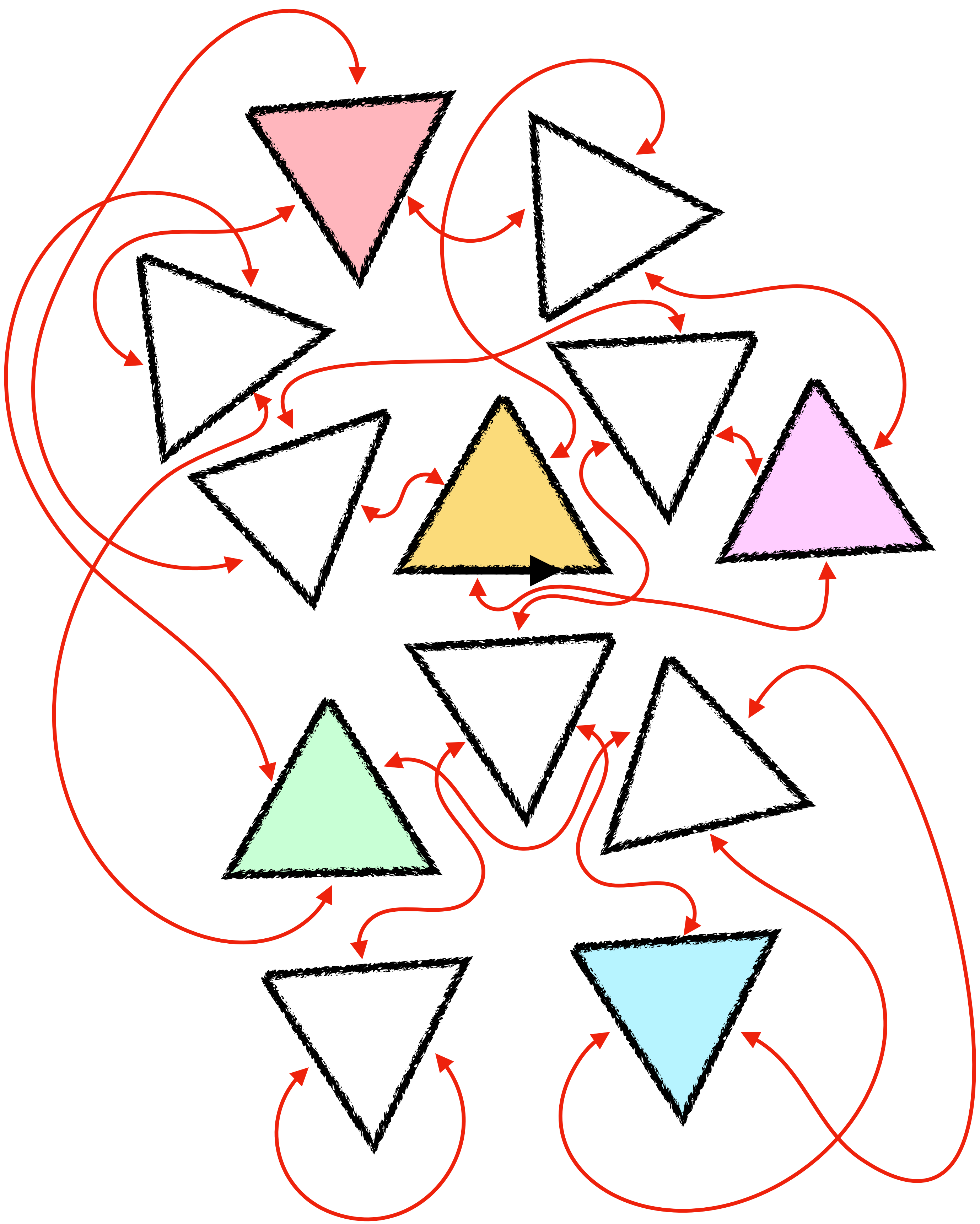


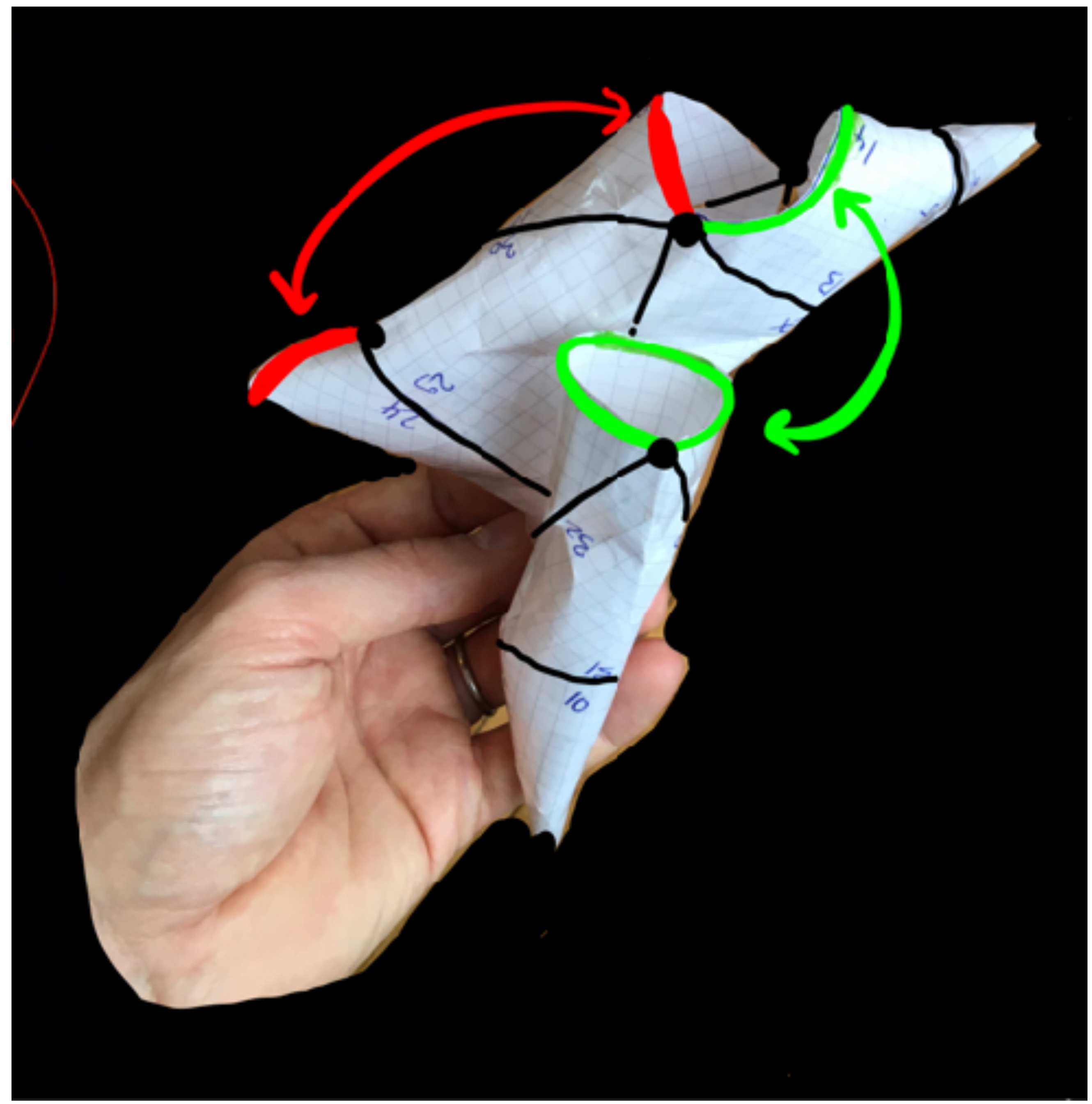
Random triangulations





Random triangulations





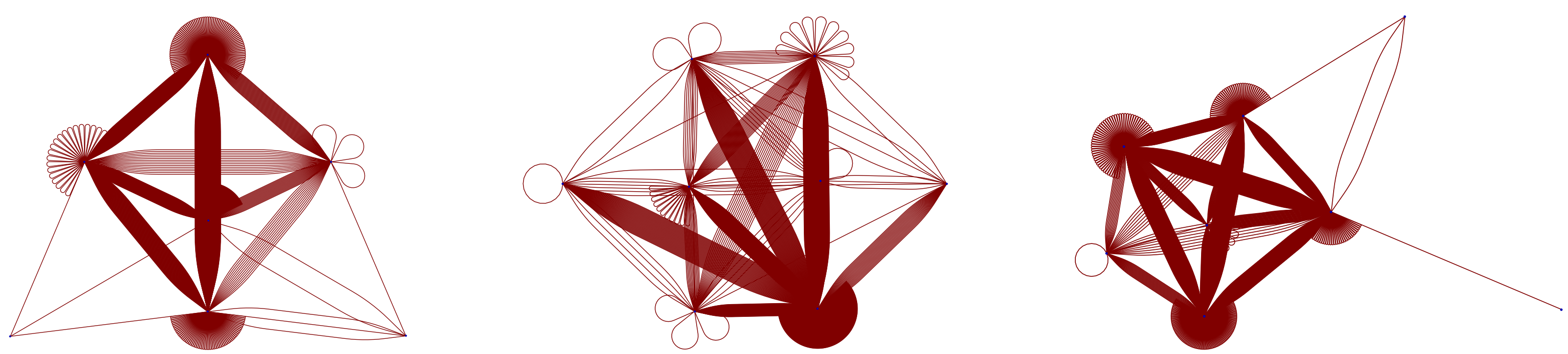
Genus 2

Random triangulations

$T_{\bullet,n}$ Uniform triangulation with n triangles

$$\mathrm{Genus}(T_{\bullet,n}) \approx \frac{n}{4} - \log n \cdot \mathcal{N}$$

Few vertices ($\log n$), high degrees (n), very small graph diameter ($O(1)$)



Brooks & Makover
Bollobas ….
Gamburd // Chmutov & Pittel
Budzinski & C. & Petri

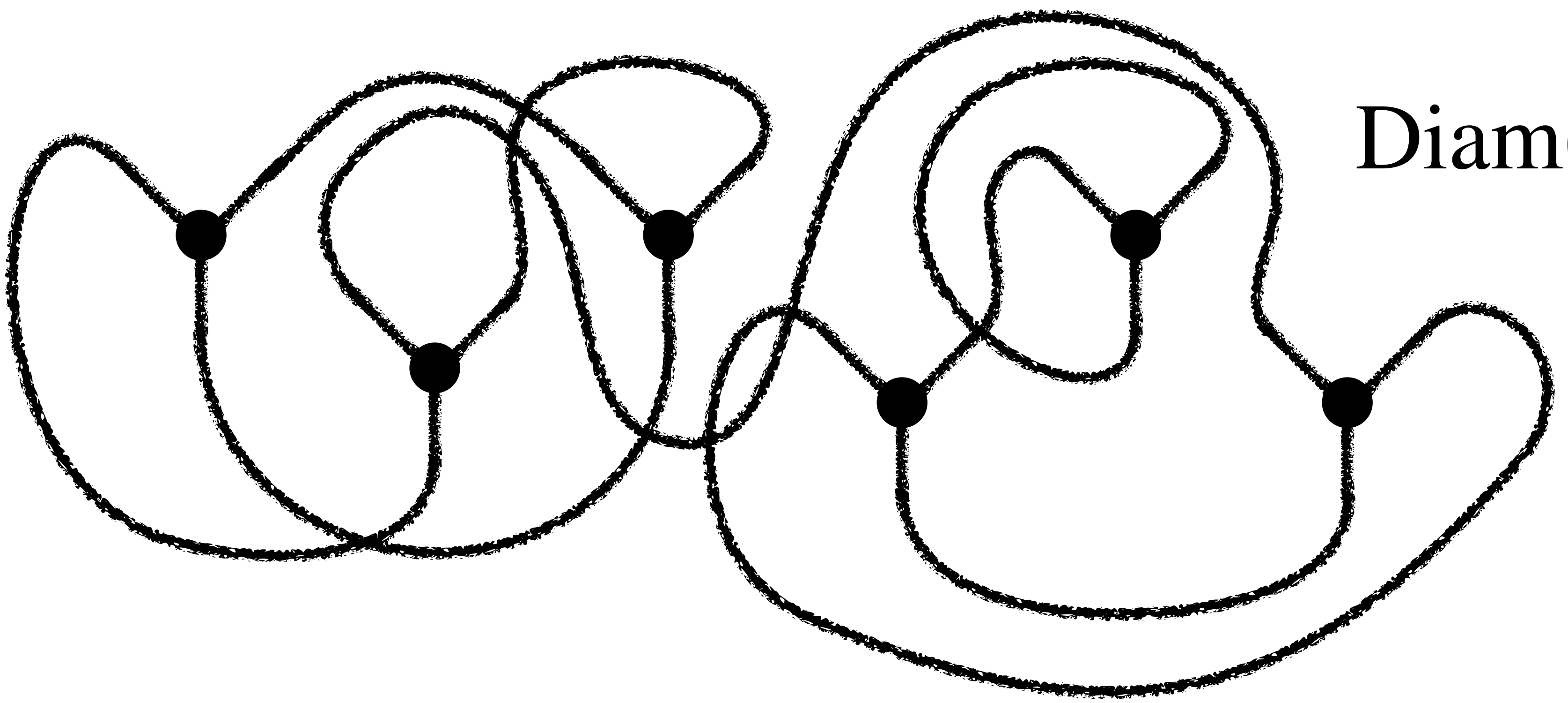
Conjecture (Budzinski, C., Petri):

The diameter of $T_{\bullet,n}$ converges in law towards an (explicit) random variable with support $\{2,3\}$.

Even more…

Configuration model

The dual of $T_{\bullet,n}$ is a configuration model or equivalently a random trivalent graph (with orientation):



$$\mathrm{Diam}(\mathrm{Dual}(T_{\bullet,n})) \approx \log_2 n$$

Bollobas & Fernandez De La Vega

If each edge of $\mathrm{Dual}(T_{\bullet,n})$ is independently given a random exponential length, then the statistics of all cycle lengths converge towards a Poisson point process (PPP) with intensity

$$\frac{\mathrm{Sinh}(t)}{t}\mathbf{1}_{t>0}\mathrm{d}t \qquad\qquad \frac{\mathrm{Cosh}(t)-1}{t}\mathbf{1}_{t>0}\mathrm{d}t$$

Janson & Louf

In the unicellular case

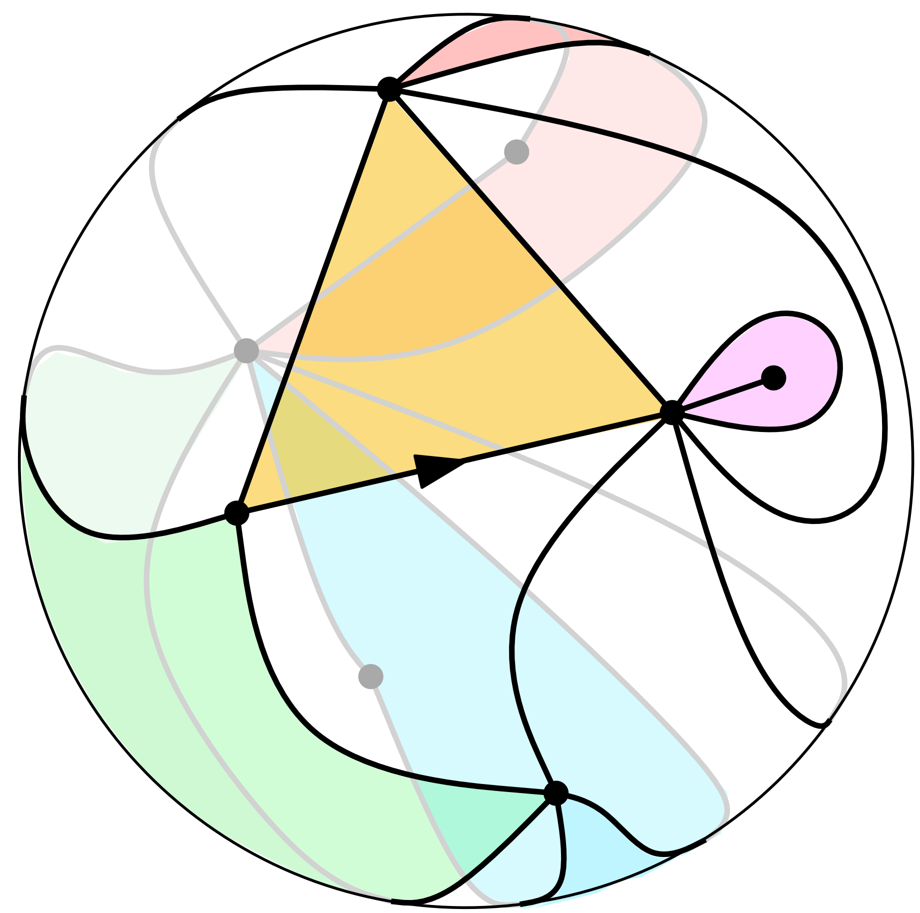
Random triangulations with constrained genus

$T_{g,n}$ uniform triangulation with n triangles and genus $0 \leq g \leq \frac{n}{4}$

$$\#\mathcal{T}_{g,n} \approx n^{2g} \exp(n \cdot f(g/n))$$

For some (rather explicit) $f : [0; 1/4] \to \mathbb{R}_+$

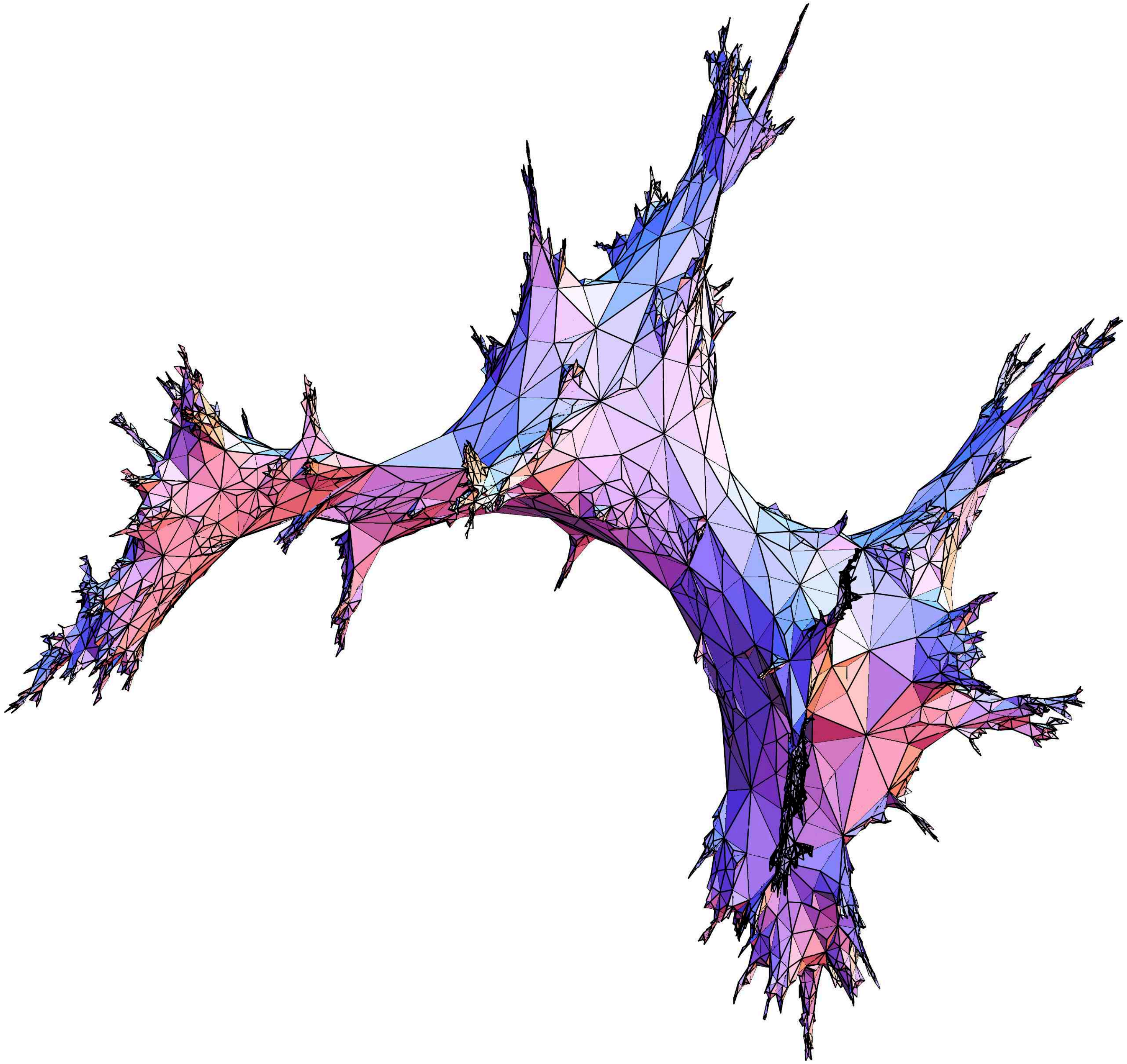
Budzinski Louf



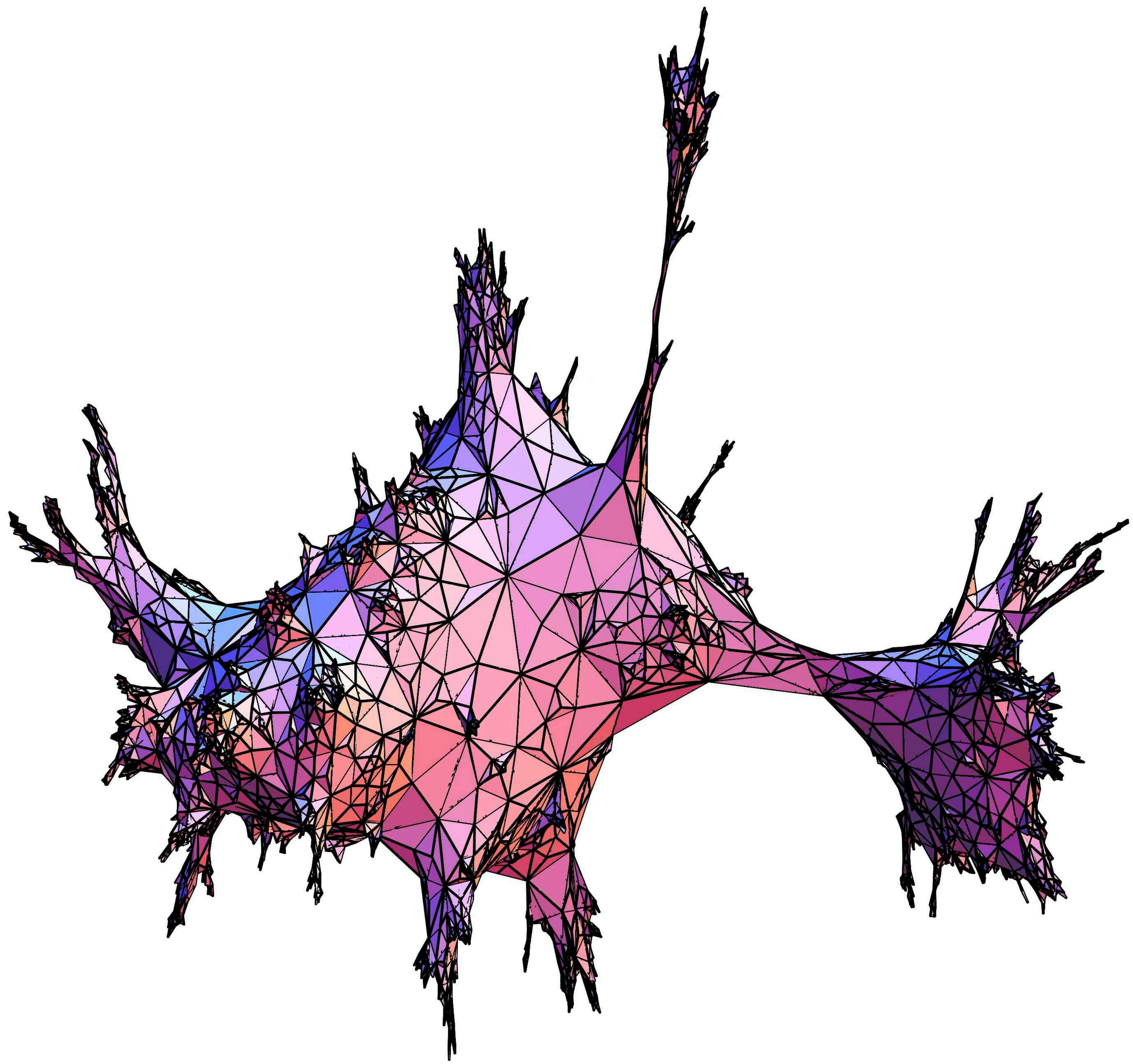
Planar case $g = 0$ (very unlikely for $T_{\bullet,n}$)

Simulations ?

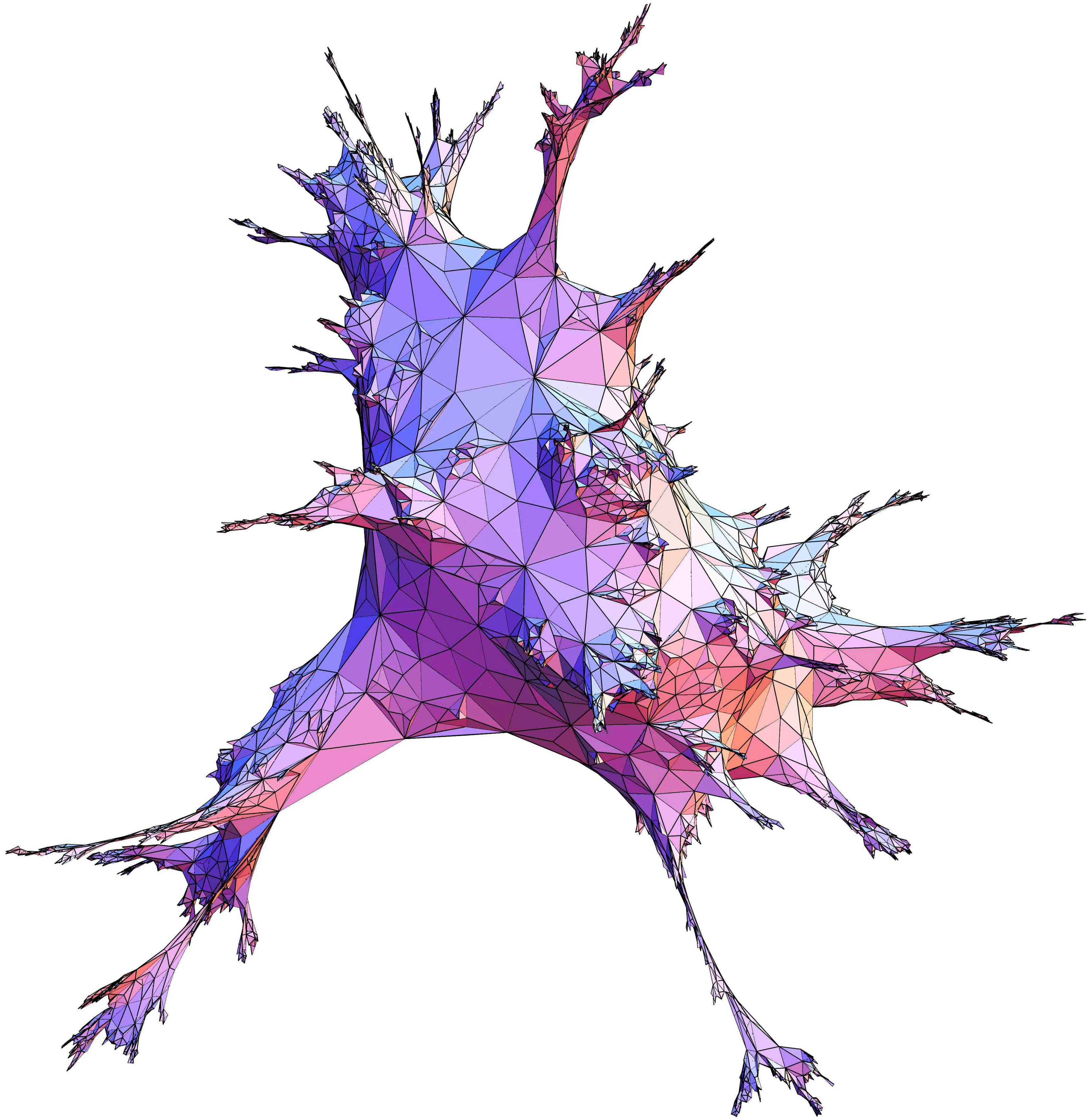
The rocking horse



The goat with an umbrella



The executioner



Période bleue

Période rose

The Brownian sphere (2011)

Theorem : We have the following convergence in law for the Gromov-Hausdorff distance on (isometry classes of) compact metric spaces:

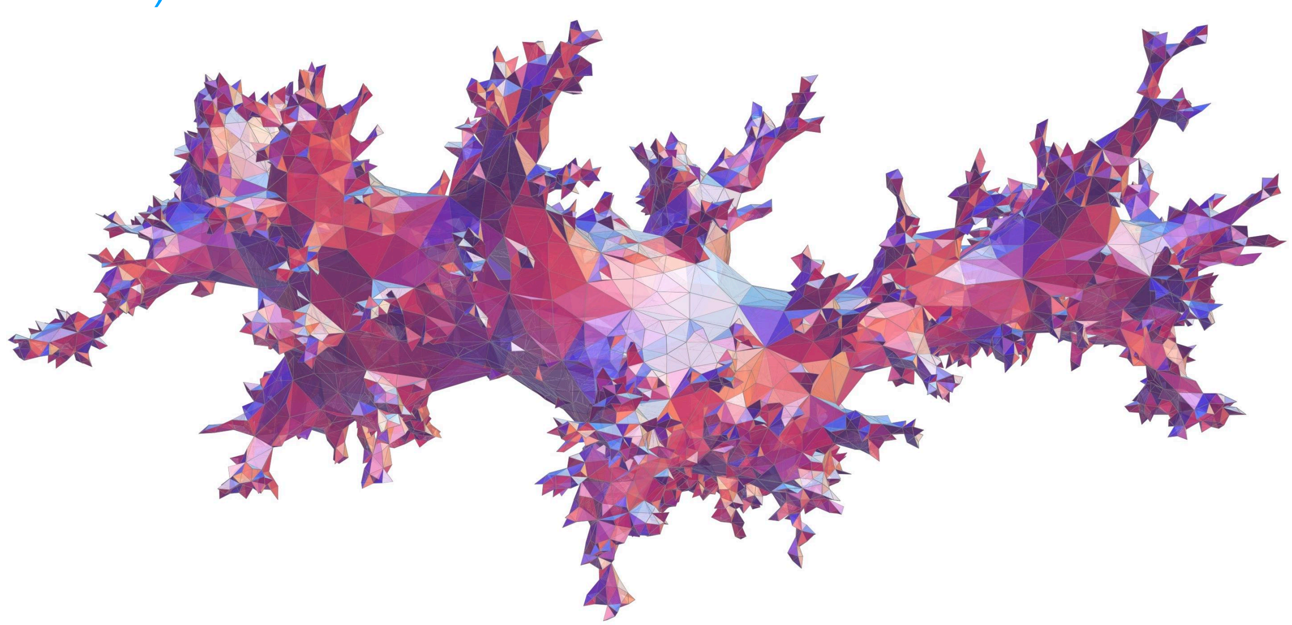
$$\left(\text{Vertices}(T_{0,n}), n^{-1/4} \cdot \mathrm{d_{gr}}\right) \xrightarrow[n\to\infty]{} (\mathbb{S}, \Delta)$$

Le Gall (see also Miermont)

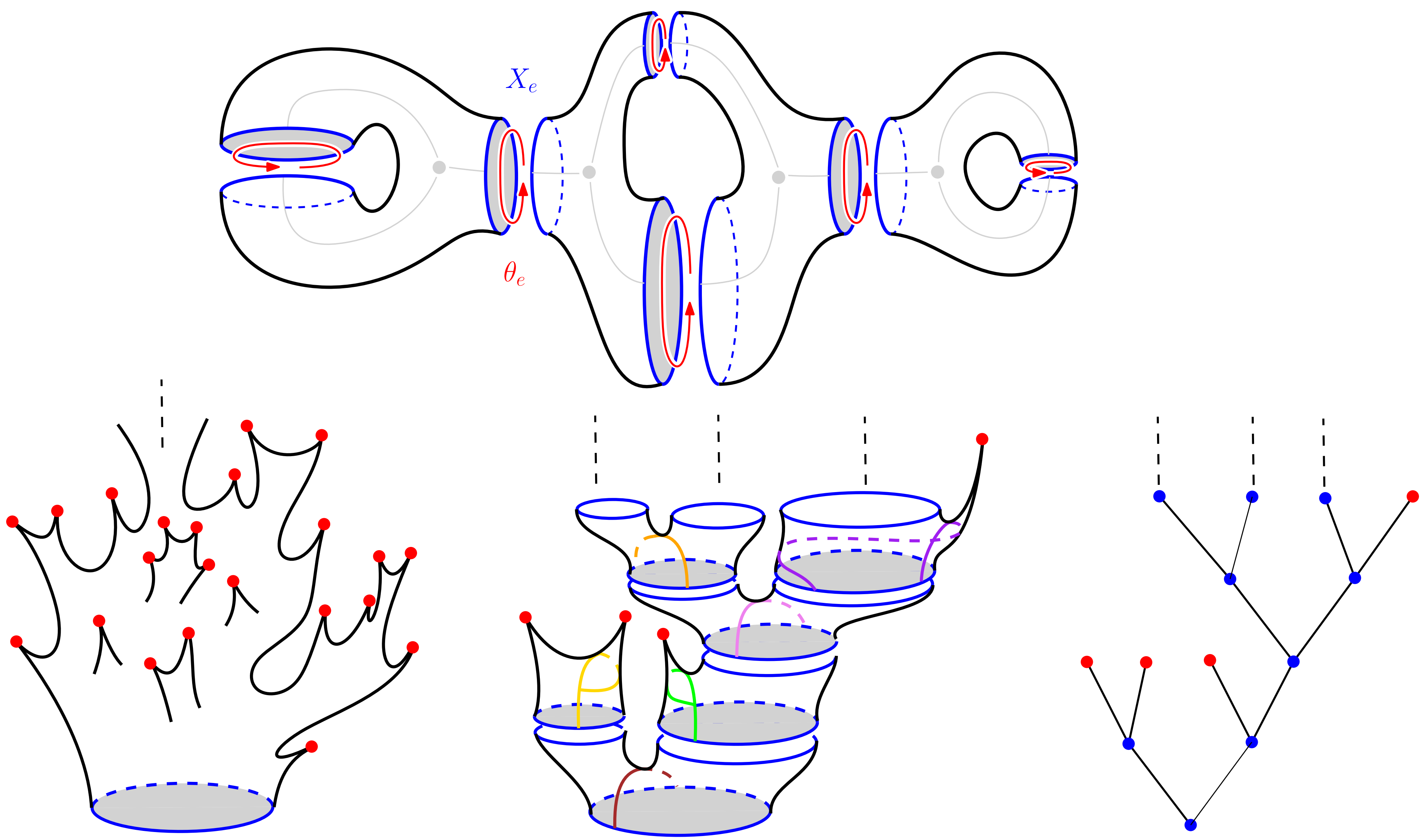
Random compact metric space

$$\dim_{\mathrm{H}}(\mathbb{S}) = 4$$

$\mathbb{S} \underset{\text{homeo}}{\sim}$ 



Part II: Random hyperbolic surfaces

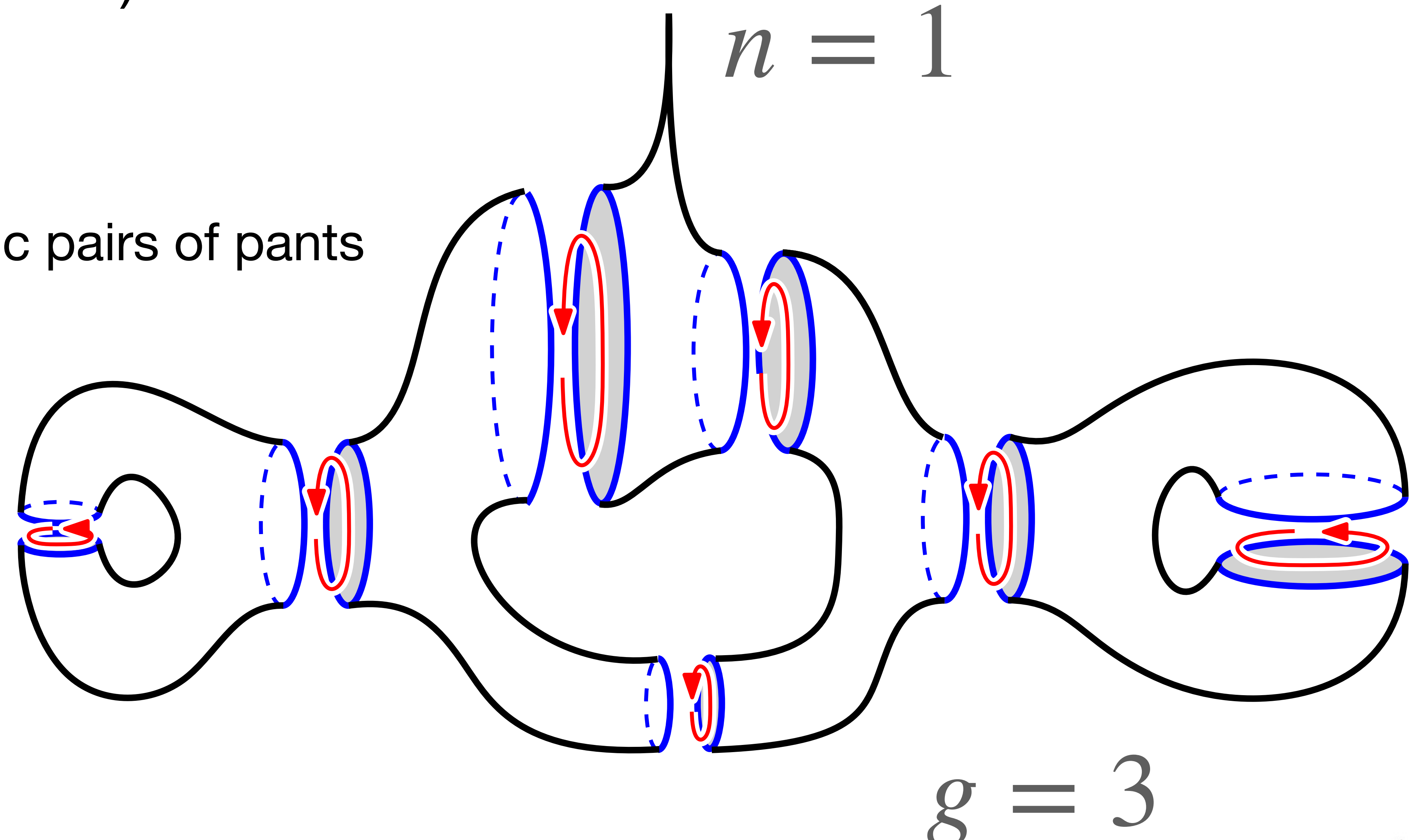


Moduli space of hyperbolic surfaces

We let $\mathscr{M}_{g,n}$ be the moduli space of isometry classes of closed hyperbolic surfaces with genus g and n punctures.

Hard to understand, usually use Teichmüller space $\mathscr{T}_{g,n}$ (overparametrized)

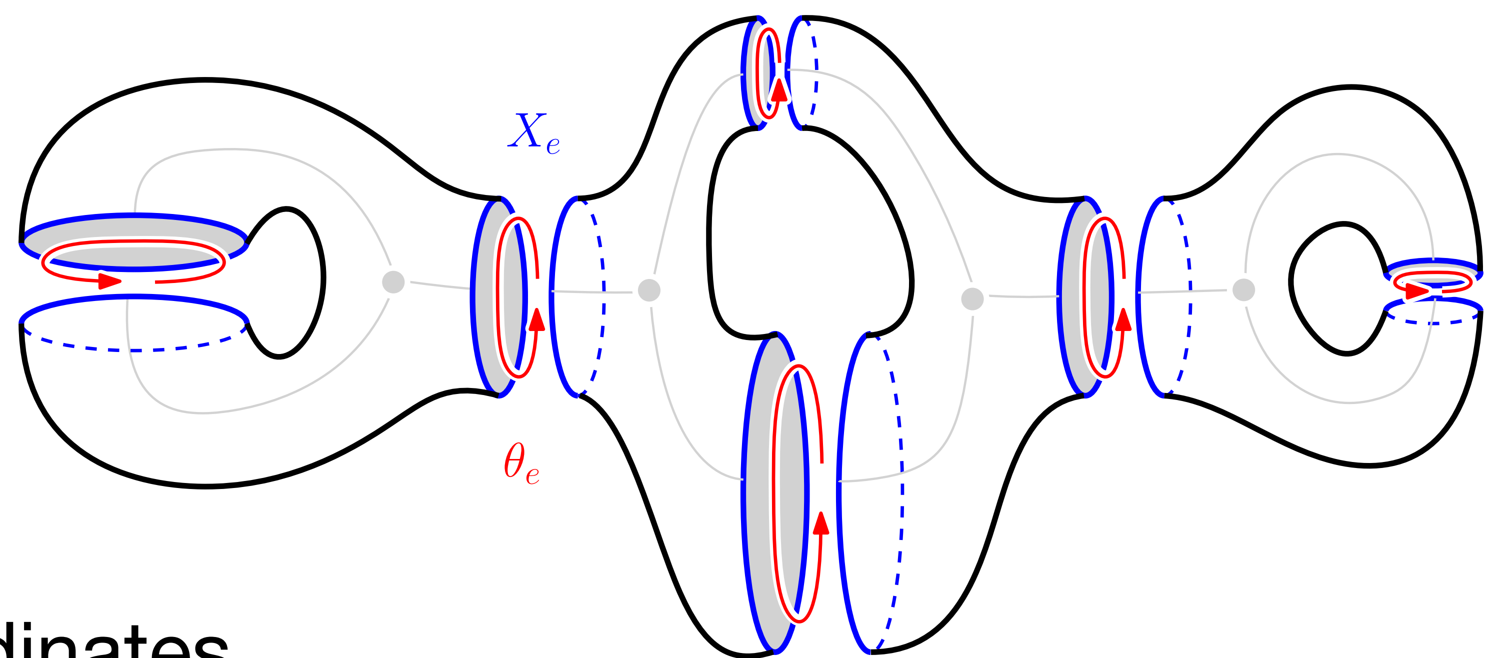
Down-to-earth:
gluing of hyperbolic pairs of pants



Weil-Peterson measure

We let $\mathscr{M}_{g,n}$ be the moduli space of isometry classes of closed hyperbolic surfaces with genus g and n punctures.

Hard to understand, usually use Teichmüller space $\mathscr{T}_{g,n}$ (overparametrized)



Fenchel-Nielsen coordinates

∞- Lebesgue measure on $\mathscr{T}_{g,n} = (\mathbb{R} \times \mathbb{R}_+)^{3g-3}$ —— Wolpert (81) —→ (MCG) Finite measure on $\mathscr{M}_{g,n}$

Random WP-hyperbolic surfaces

We $\mathcal{S}_{g,n}$ be a random hyperbolic surface sampled according to (normalized) WP measure on $\mathcal{M}_{g,n}$

In high genus as $g \to \infty$

The random surface $\mathcal{S}_{g,0}$ has logarithmic diameter, a spectral gap, a positive Cheeger constant… (non optimal constants)

Mirzakhani

+

Petri

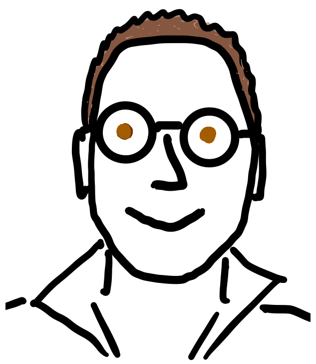
Set of lengths of primitive closed loops on $\mathcal{S}_{g,0}$ converges in law towards a PPP with intensity

$$\frac{\mathrm{Cosh}(t) - 1}{t} \mathbf{1}_{t>0} \mathrm{d}t$$

Other works (Guth, Parlier, Young) especially recently on the spectral gap

Random WP-hyperbolic surfaces

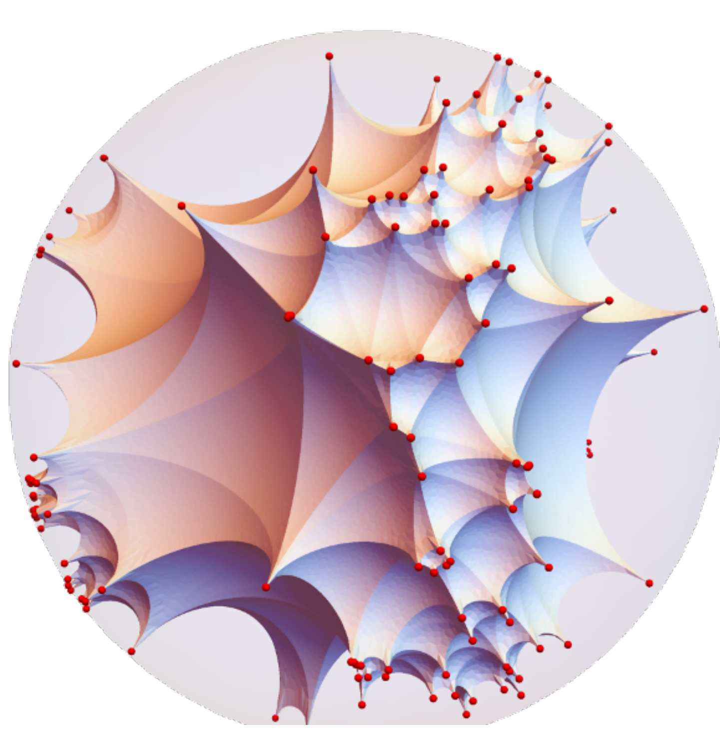
T. Budd

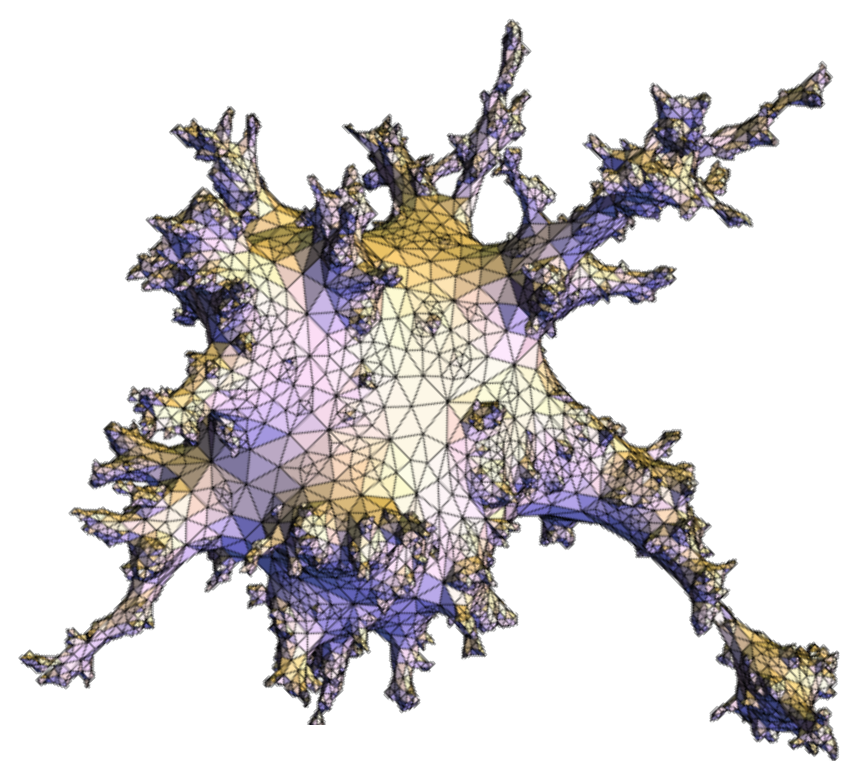


In genus 0 with many punctures we have the convergence in distribution

$$n^{-1/4} \cdot \mathcal{S}_{0,n} \longrightarrow \text{Brownian Sphere}^*$$

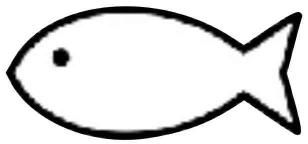
As $n \to \infty$ for the Gromov—Prokhorov distance**





Courtesy of Timothy Budd

There's something fishy about it, isn'it ?



Two must useful tools

Tree bijections

Peeling process

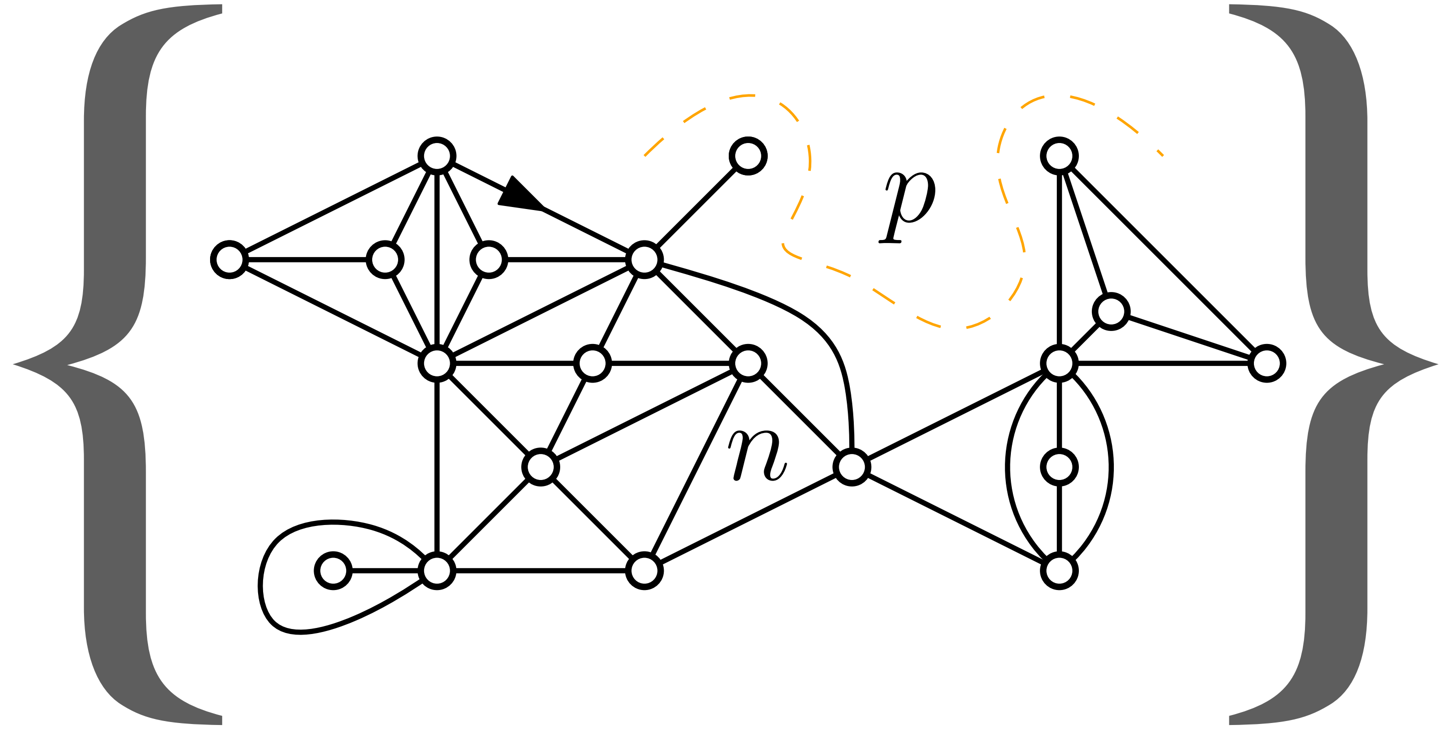
Tableau 2: Tutte, Mirzakhani and

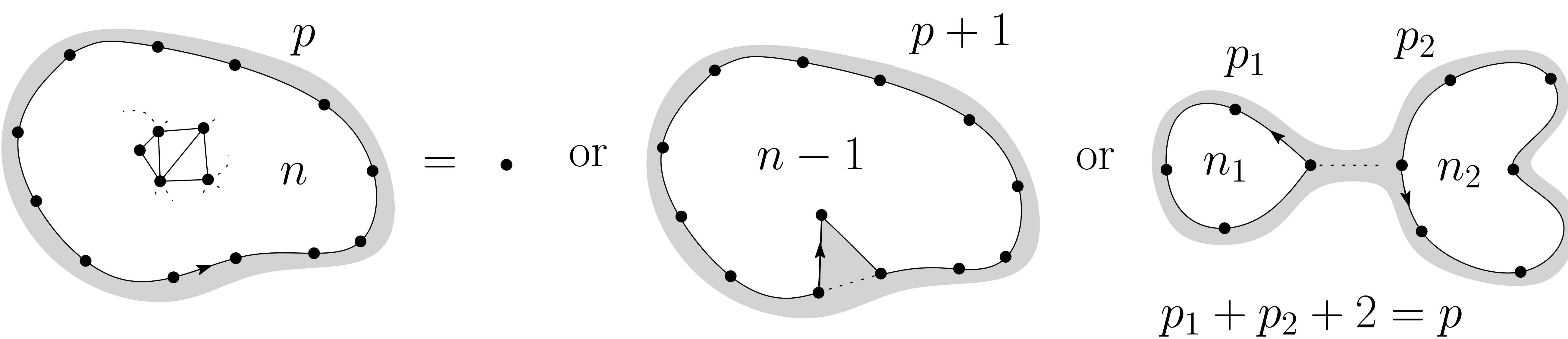
Peeling process

Ongoing work with Thomas Budzinski & Bram Petri

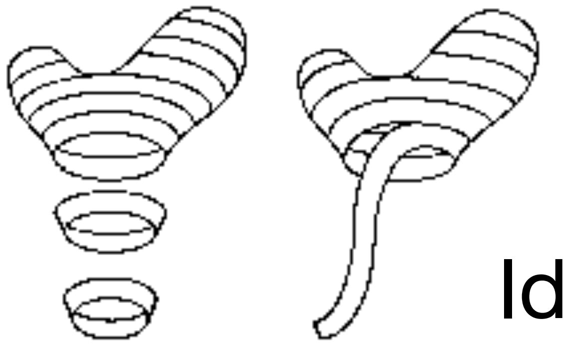
Tutte's equation for triangulations

$$\mathbb{T}^{(p)}_{0,n} = \#$$



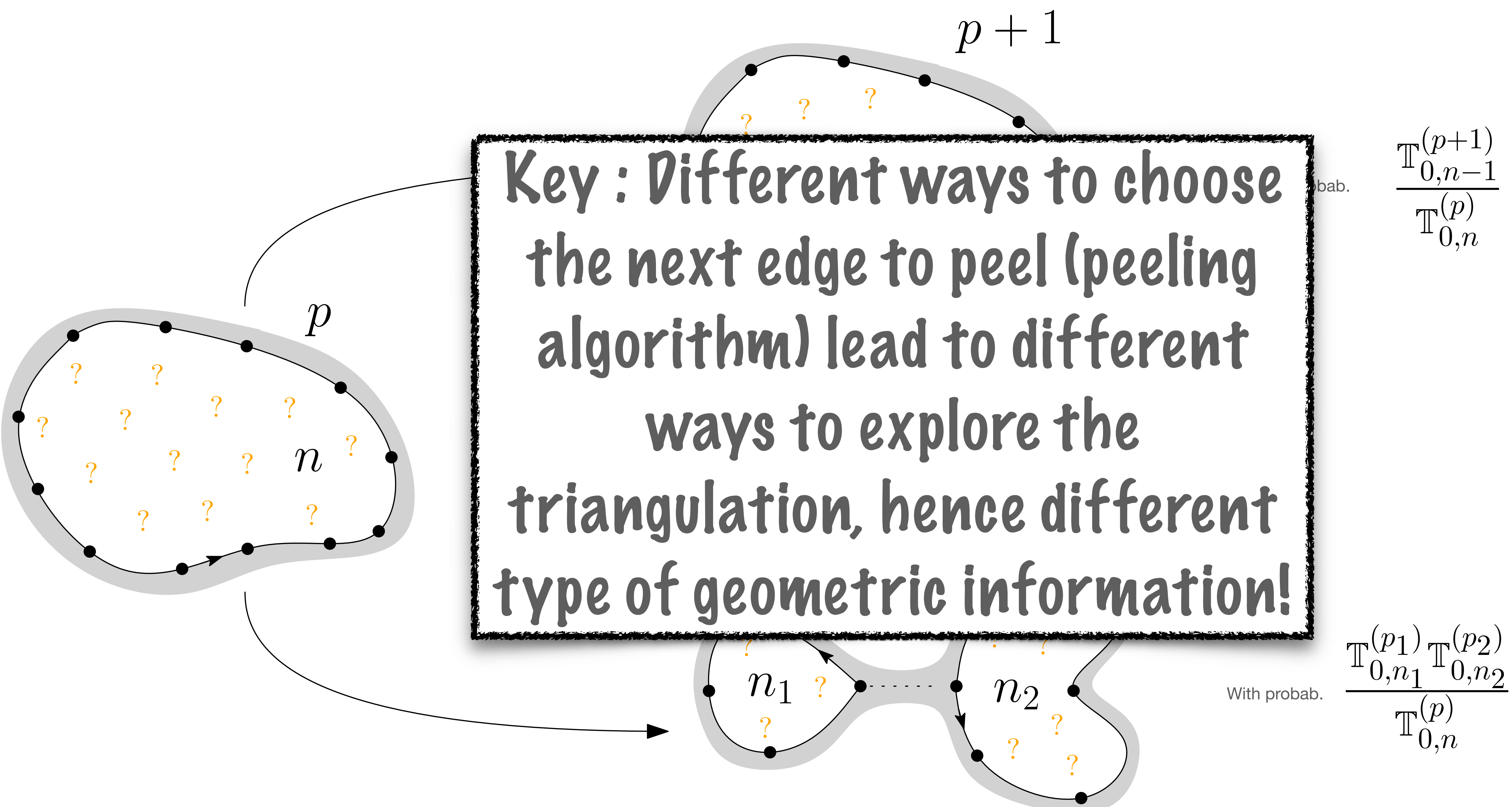


Recursion on $\mathbb{T}^{(p)}_{0,n}$



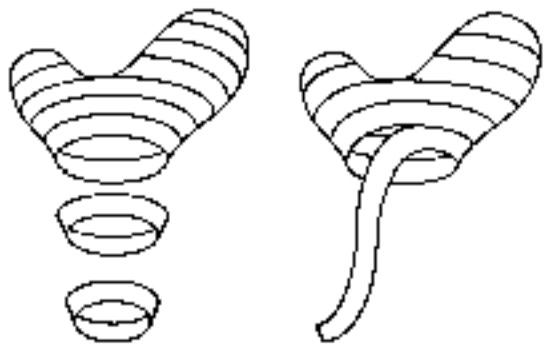
Peeling process

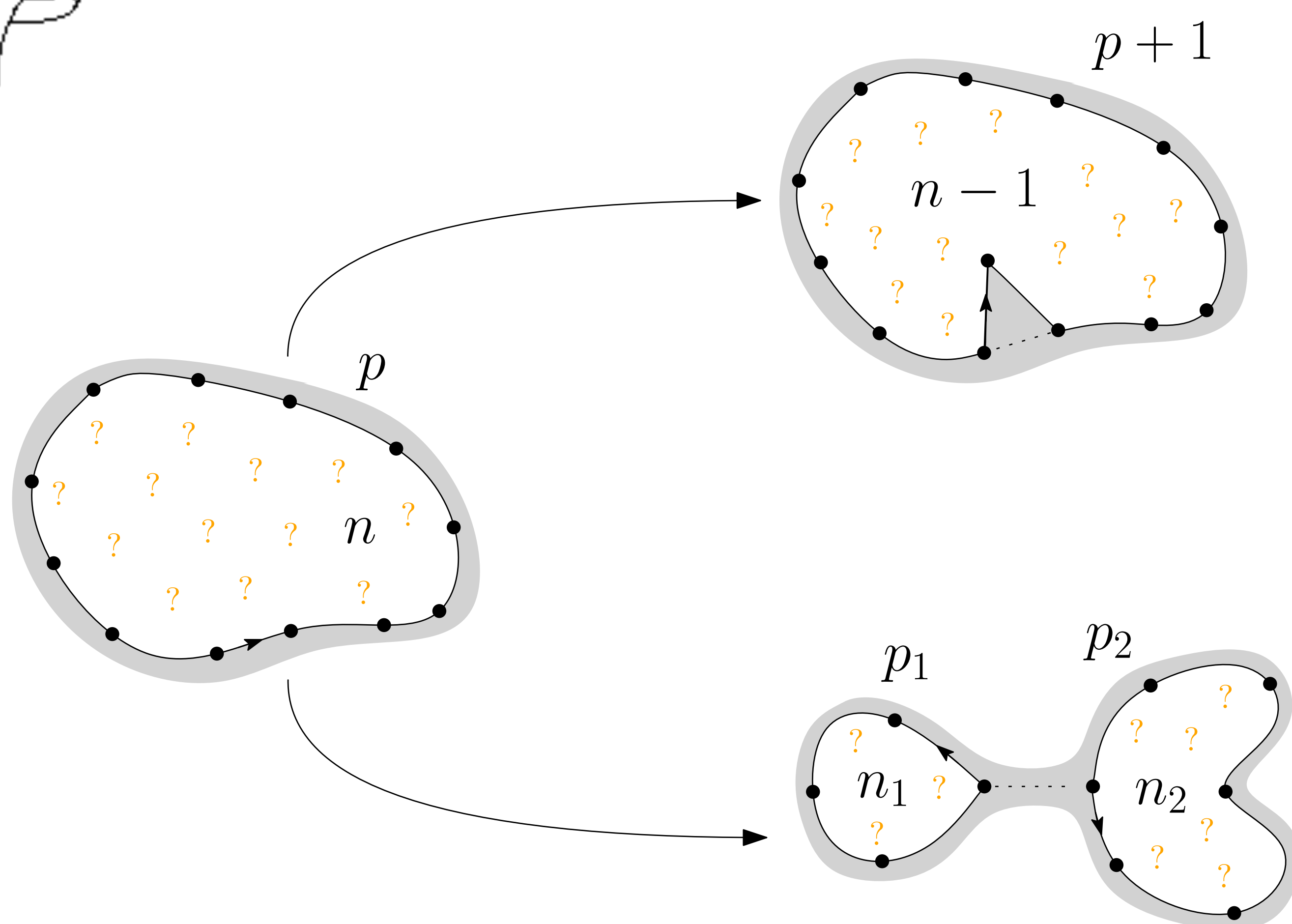
Idea: turn Tutte's equation into a growth process/ exploration mechanism of random triangulations.



Key : Different ways to choose the next edge to peel (peeling algorithm) lead to different ways to explore the triangulation, hence different type of geometric information!

Peeling process



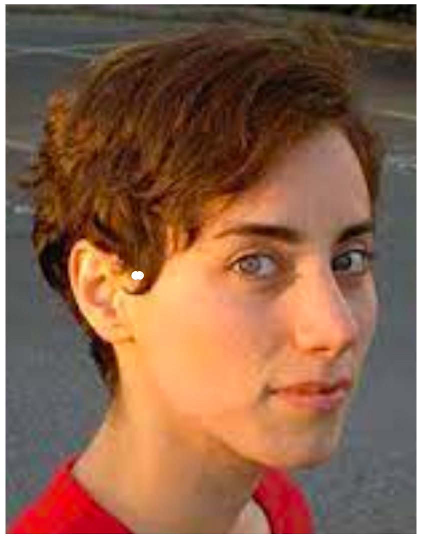


$$\frac{\mathbb{T}^{(p+1)}_{0,n-1}}{\mathbb{T}^{(p)}_{0,n}}$$

$$\frac{\mathbb{T}^{(p_1)}_{0,n_1}\mathbb{T}^{(p_2)}_{0,n_2}}{\mathbb{T}^{(p)}_{0,n}}$$

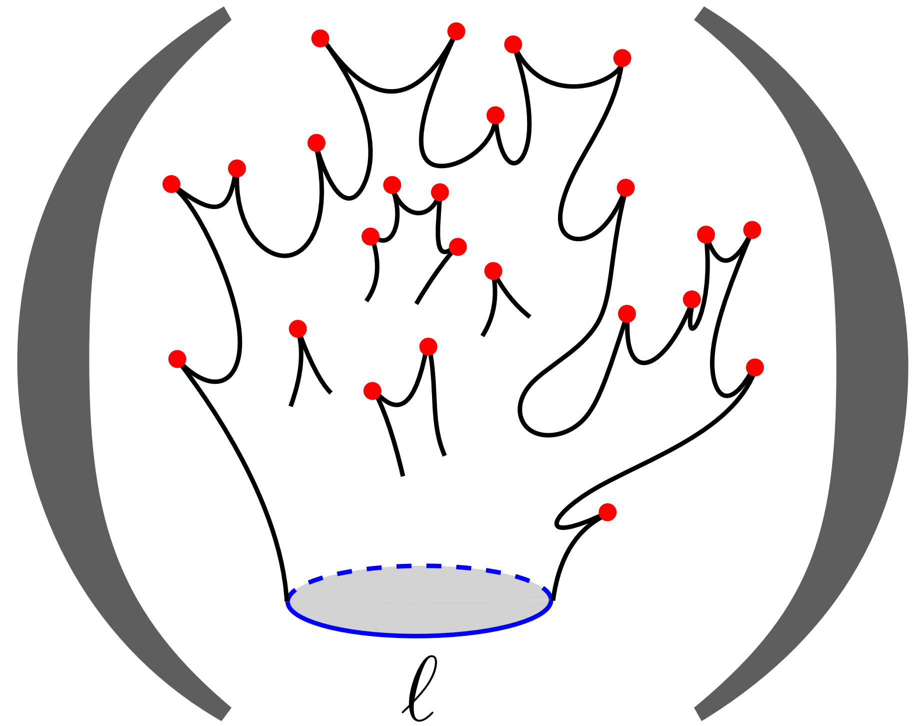
Applications :

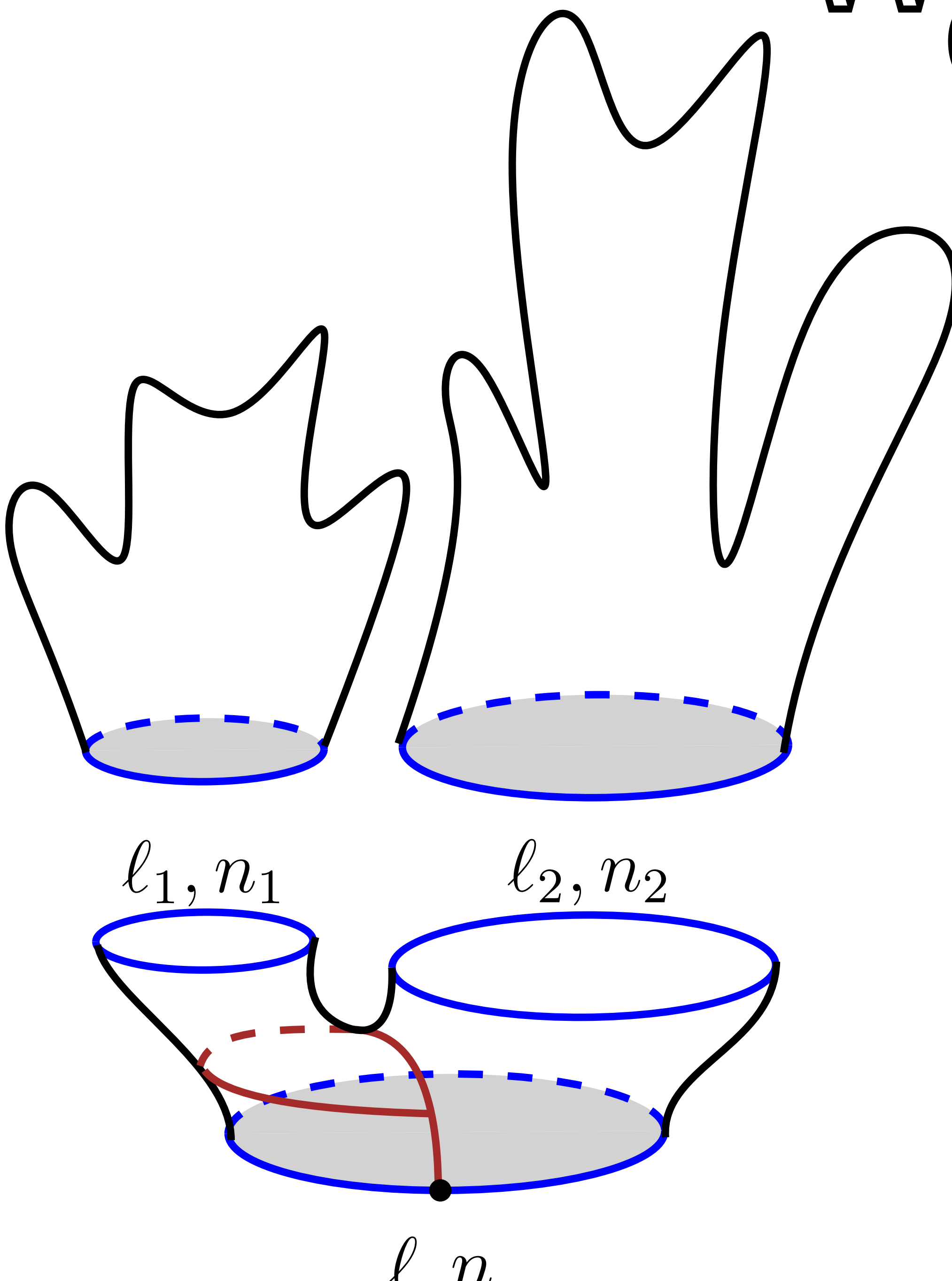
- Study the volume growth (recovering $n^{1/4}$ diameter)
  Angel, C. & Le Gall
- Study the behavior of simple random walk
  Benjamini & C.
- Study Bernoulli percolation
  Angel, Angel & C., C. & Richier, Budd & C.



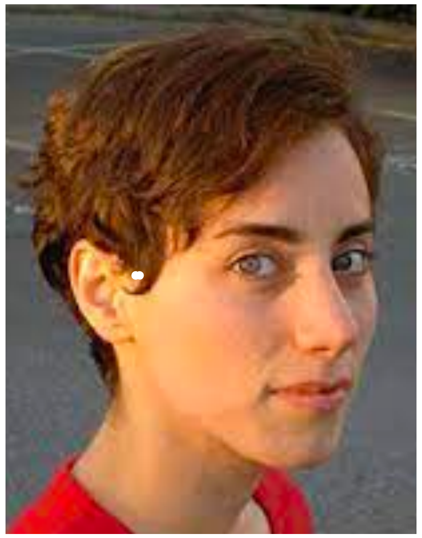
Mirzakahni's recursion for WP volumes

$$\mathbb{W}^{(\ell)}_{0,n} = \mathrm{WP}\Bigg(\quad\Bigg)$$



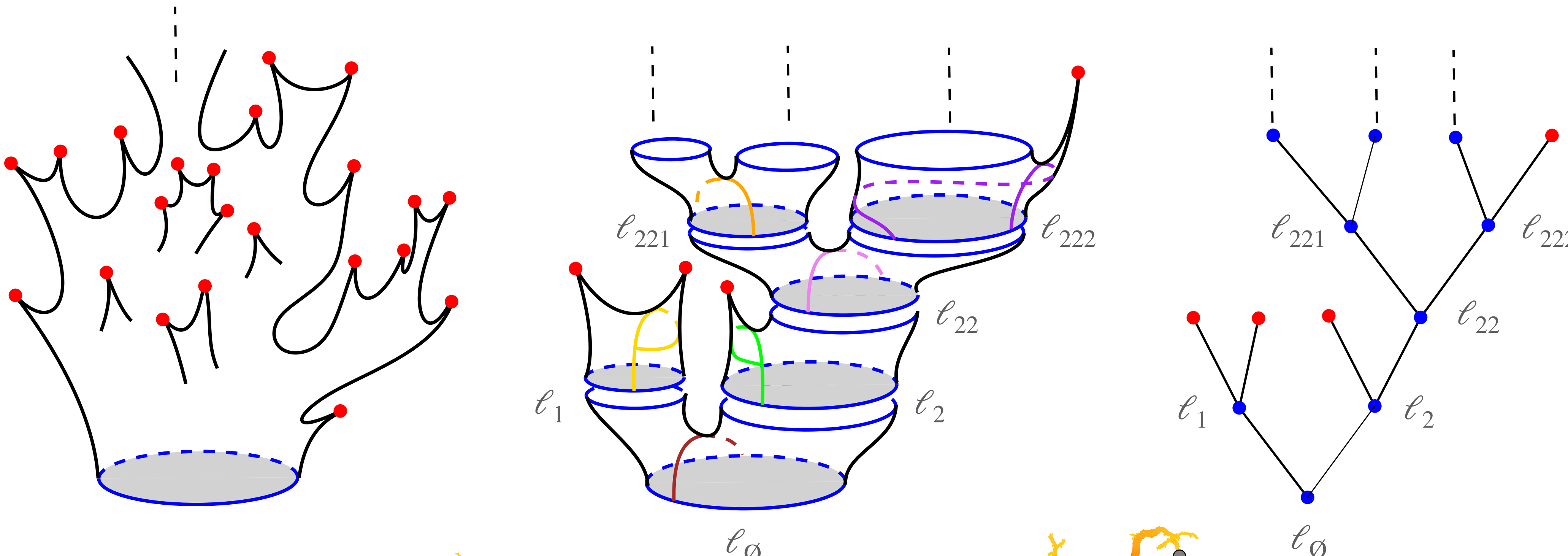


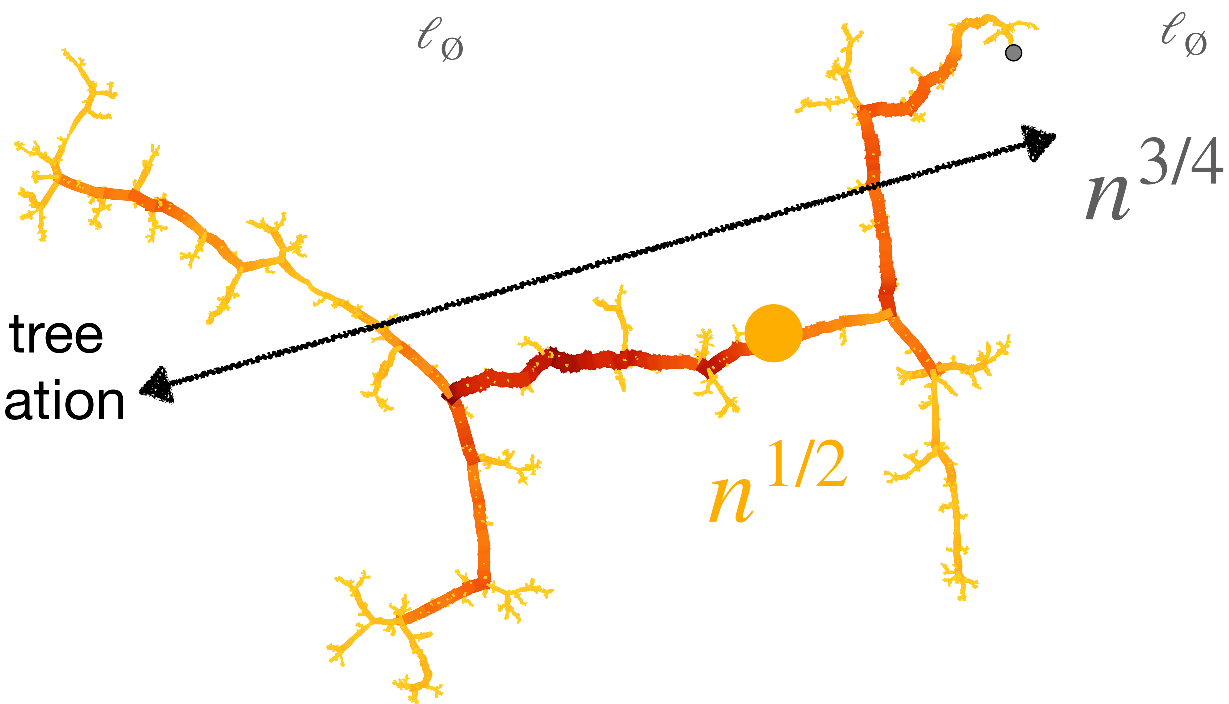
Recursion on $\mathbb{W}^{(\ell)}_{0,n}$



Turn Mirzakani's recursion into peeling

Develop the peeling process of WP surfaces. In genus 0 in particular, connect to the Growth-Fragmentation trees introduced recently by Bertoin.

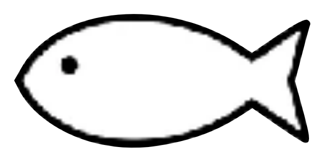




Same law of random labeled tree
But different geometric information

Teasing for Budzinski, C., Petri 2022++

There's something fishy about it, isn'it ?

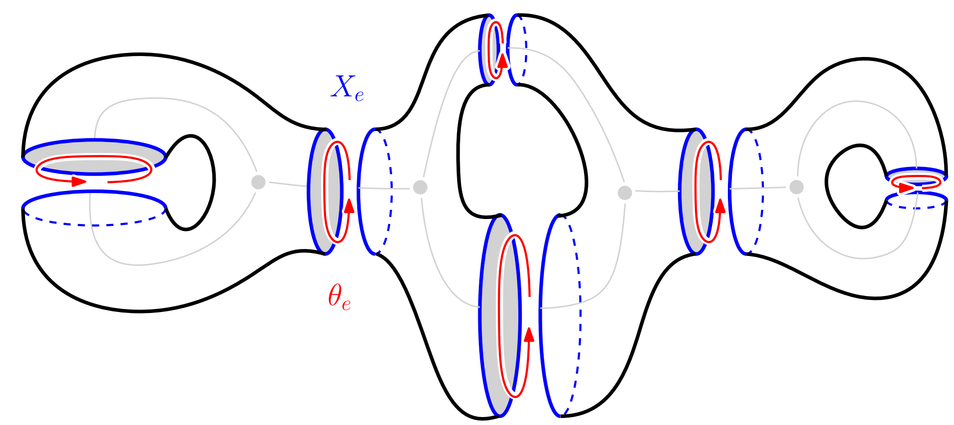


Section from $\mathscr{M}_{g,n}$ to $\mathscr{T}_{g,n}$

Fenchel-Nielsen coordinates

$$\mathbb{R}^{3g-3} \times \mathbb{R}_+^{3g-3} = \mathscr{T}_{g,n} \twoheadrightarrow \mathscr{M}_{g,n}$$

Overparametrized



By imposing geometric constraint on the pant decomposition

(In the case of Mirzakhani, that the pairs of pants are compatible with the launchs of geodesics)

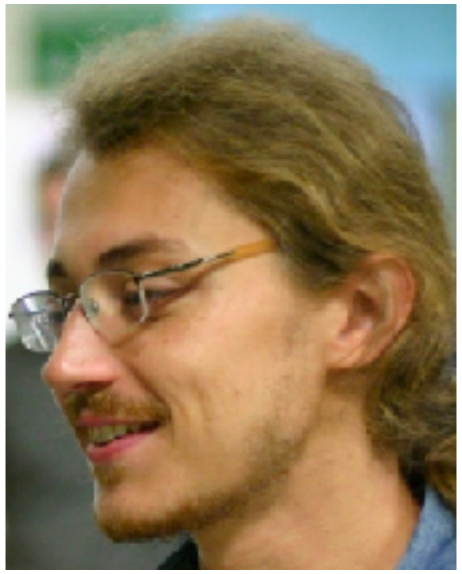
One can choose unique coordinates representing a given hyperbolic surface.

Tableau 3: Penner, Schaeffer and

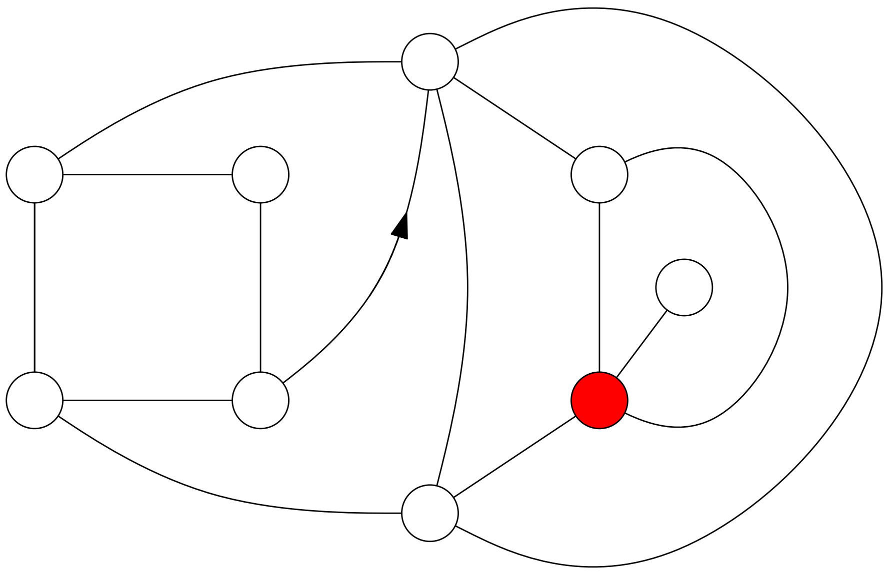
Tree bijections

Ongoing work with Timothy Budd

Schaeffer-type constructions

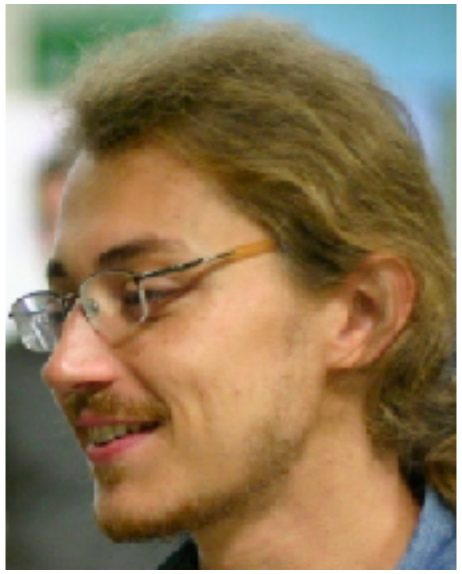


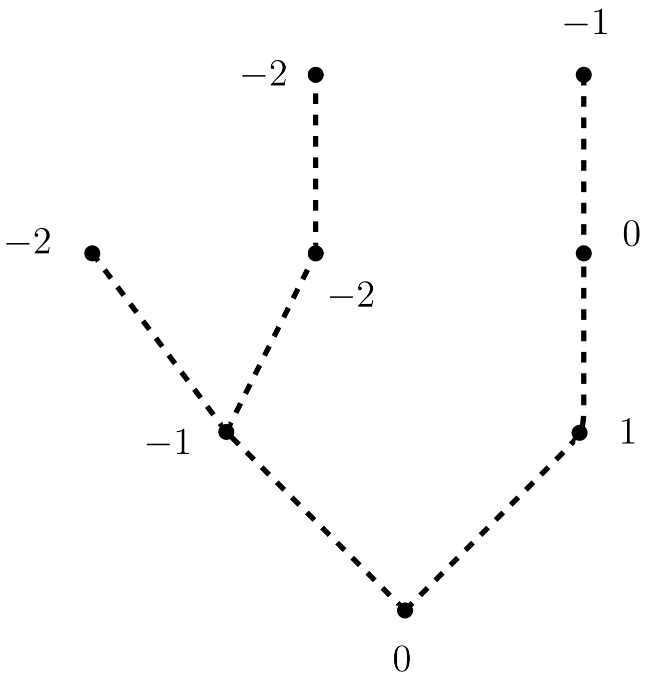
Quadrangulation



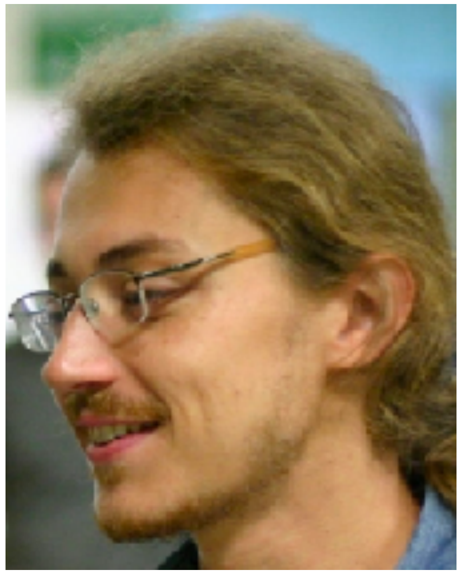
Encoding via a *labeled* tree where labels represent ***Distances*** from the distinguished red point.

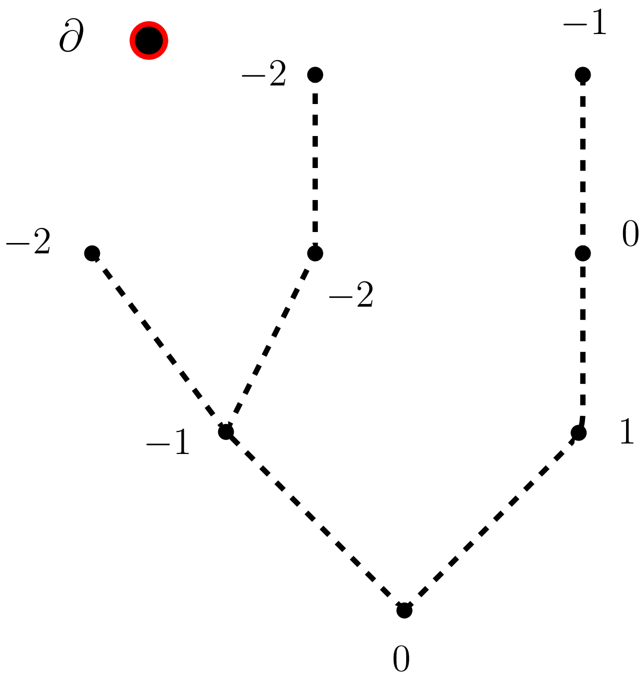
Schaeffer-type constructions





Schaeffer-type constructions





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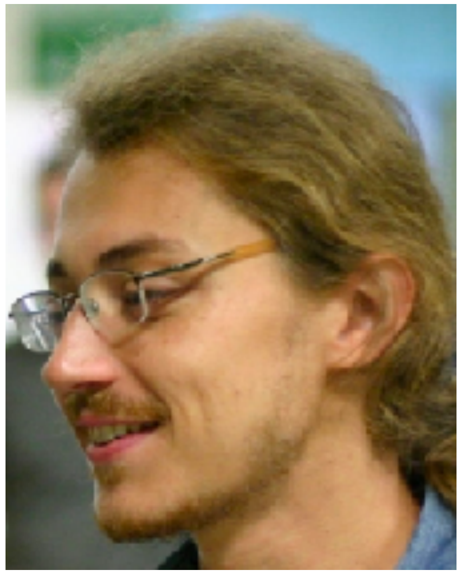
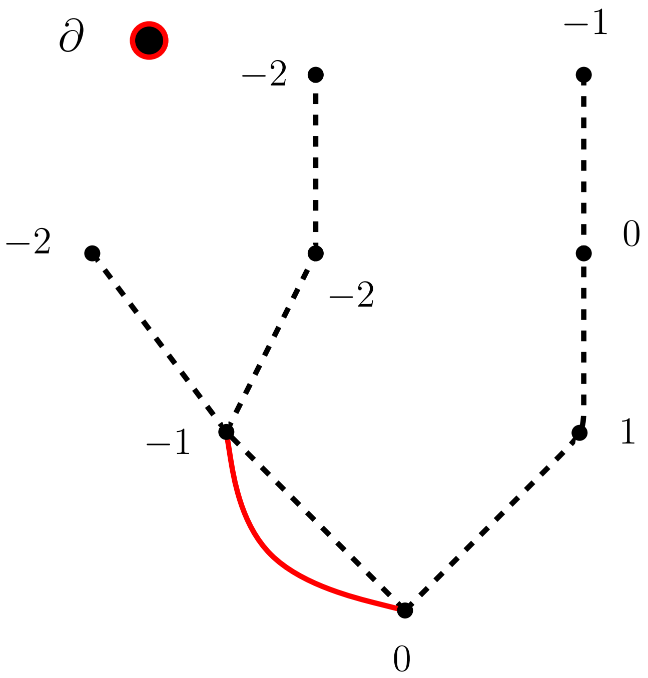
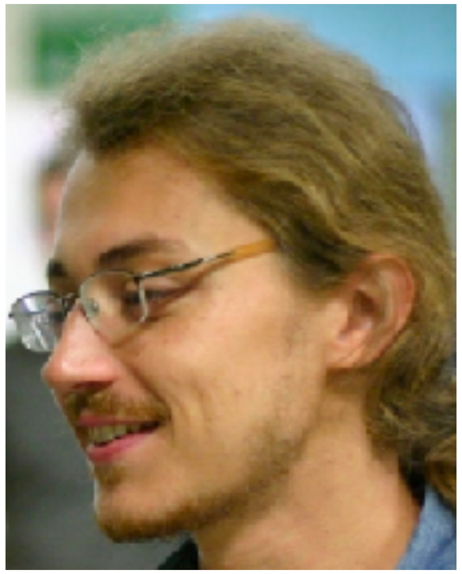
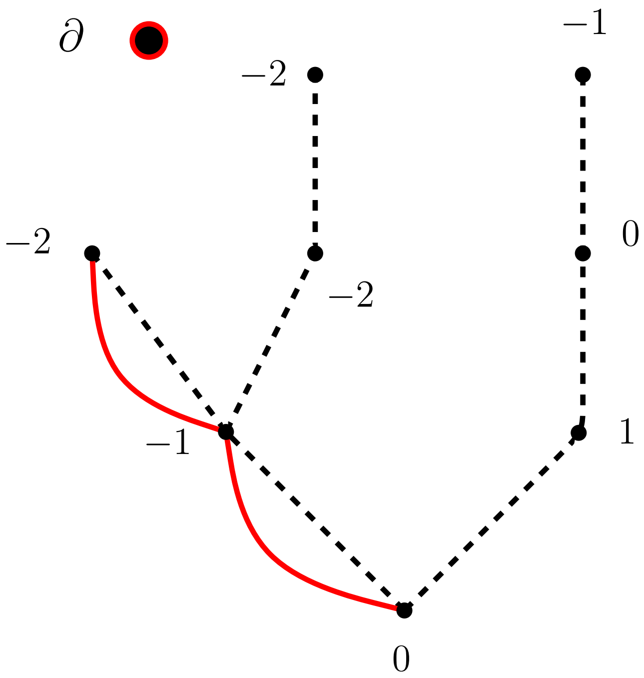




Schaeffer-type constructions





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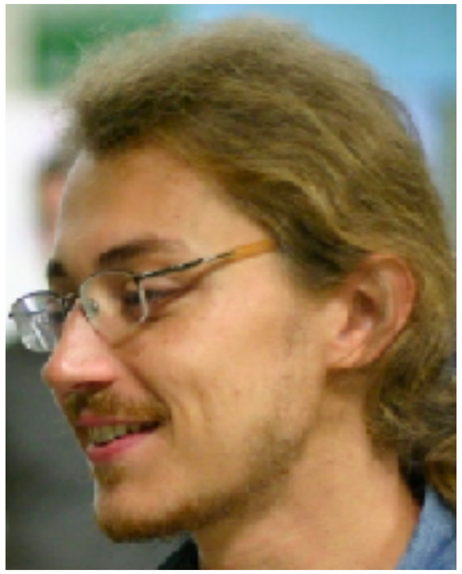

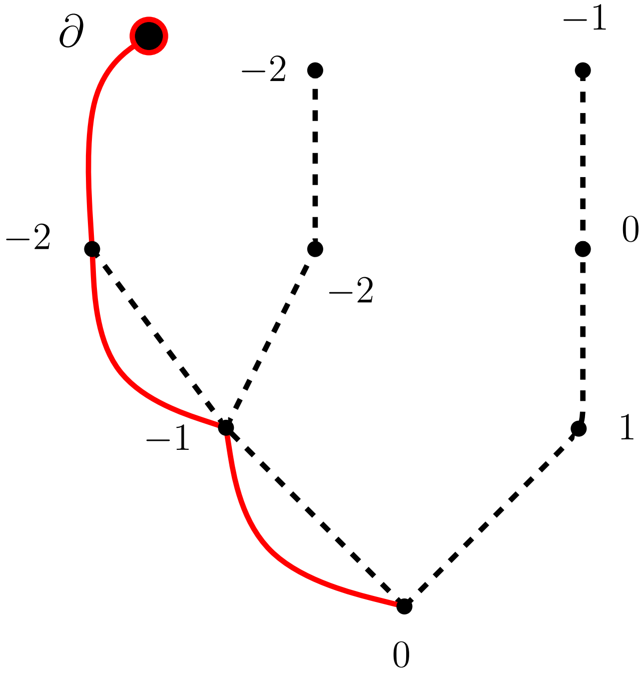

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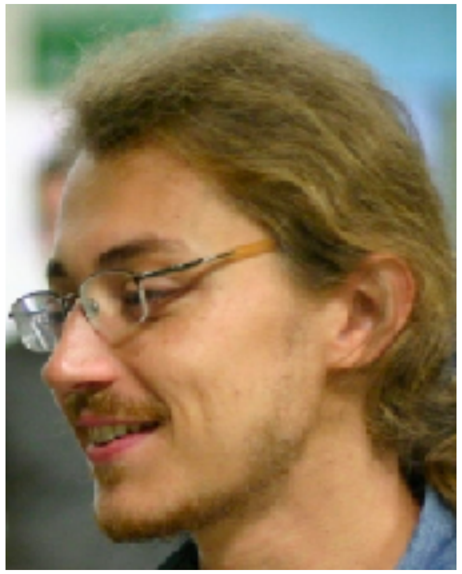

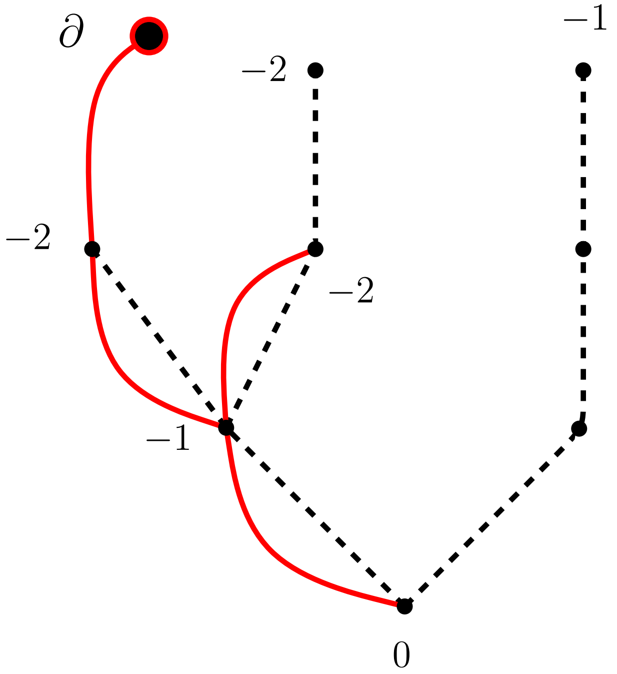

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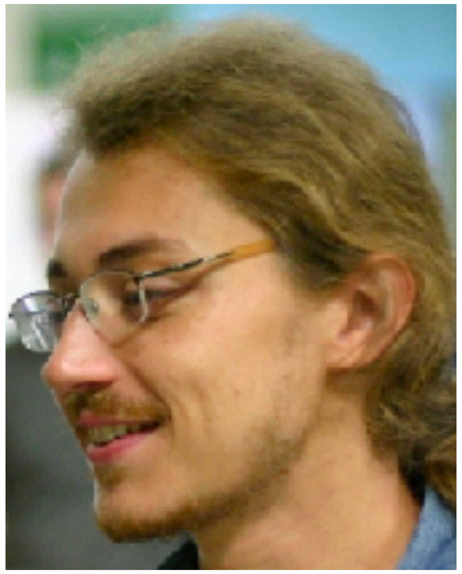

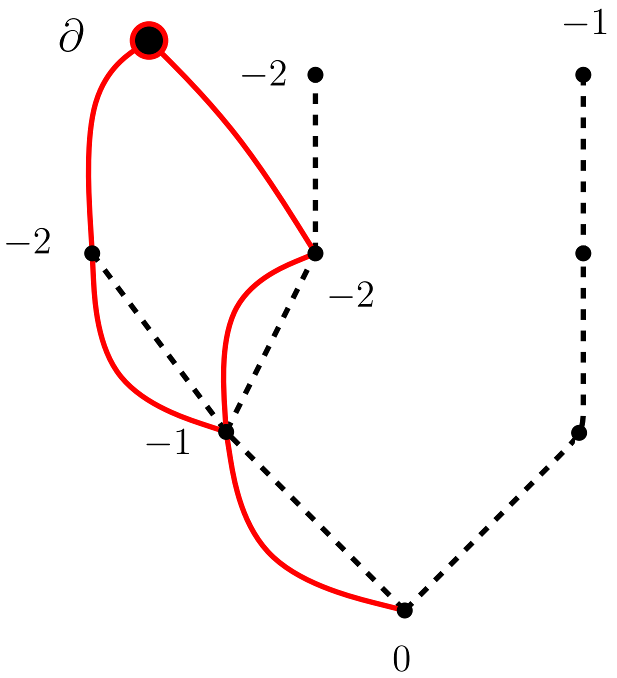

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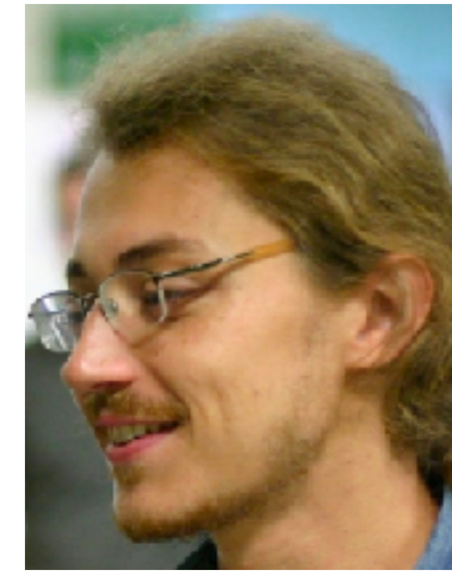

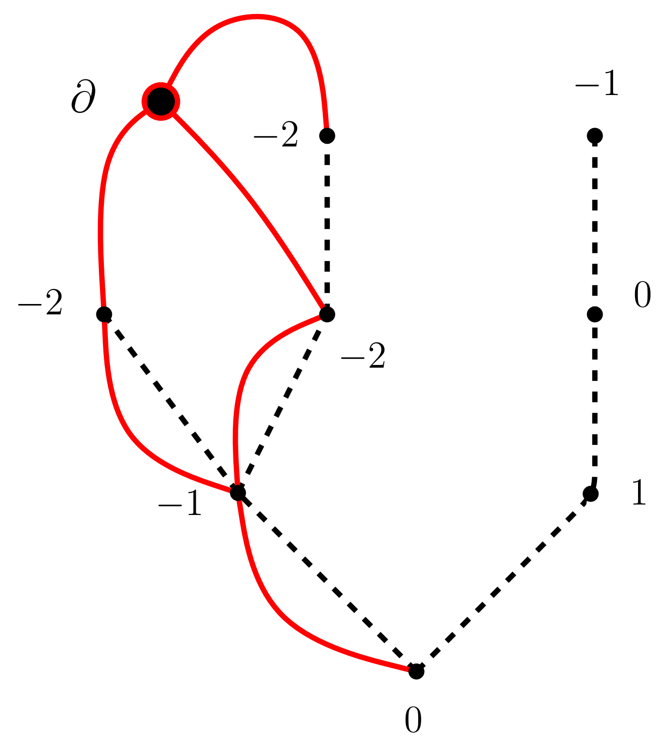

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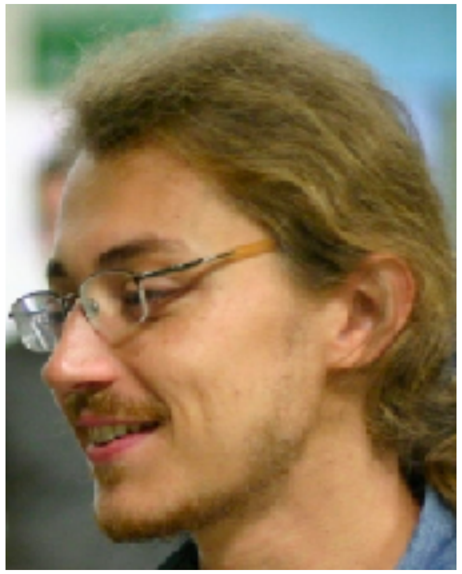
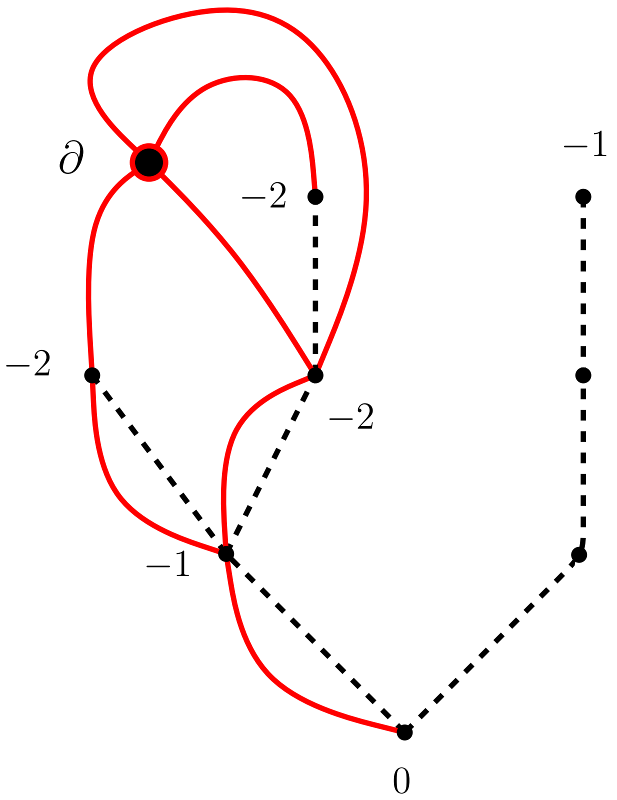
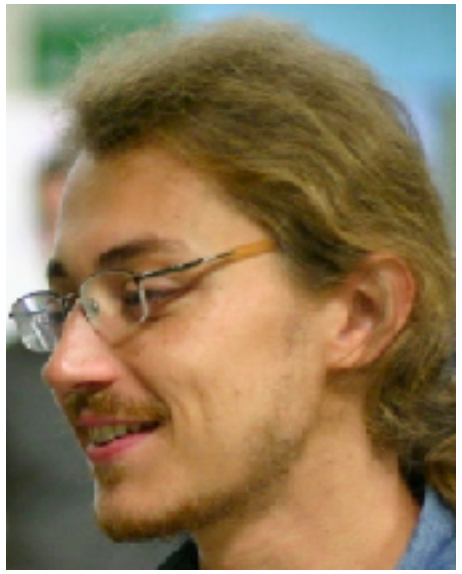
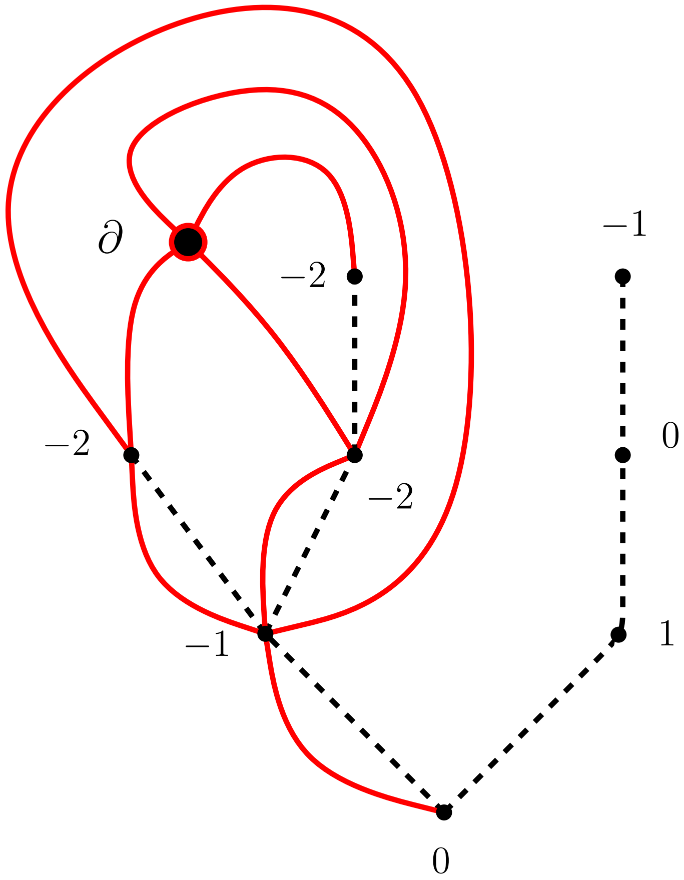


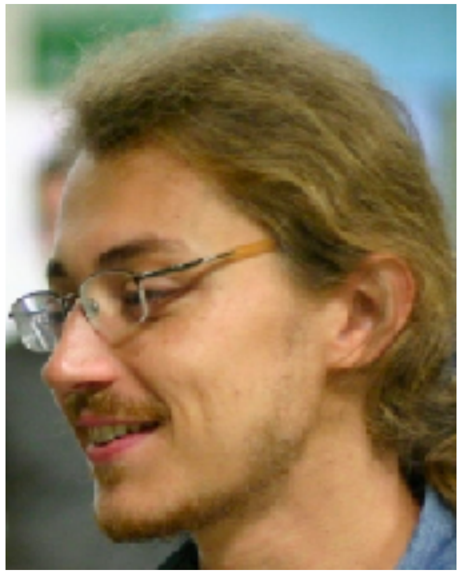


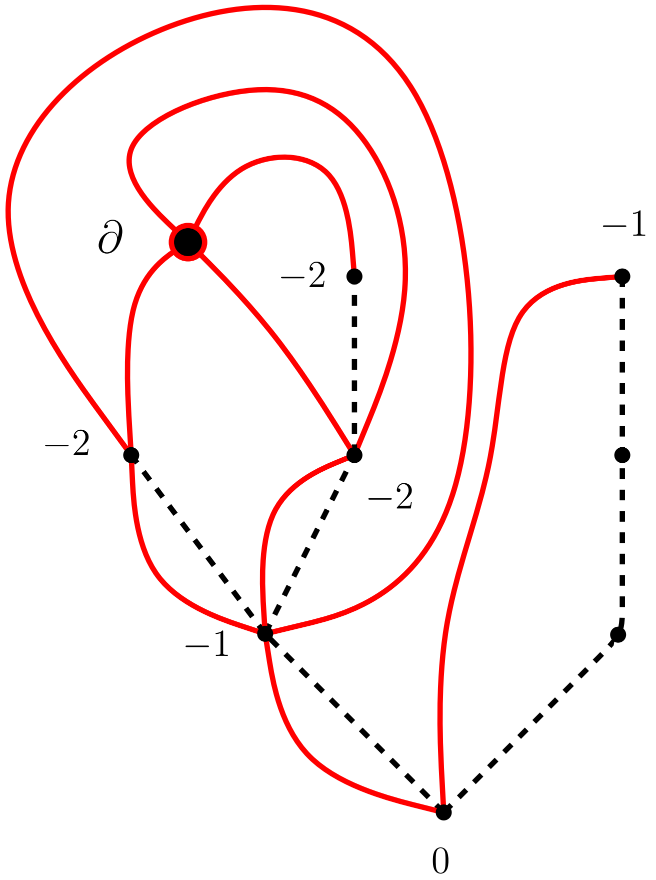
Schaeffer-type constructions



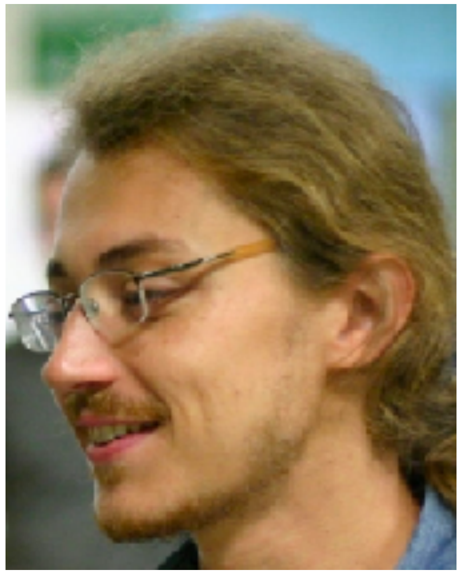


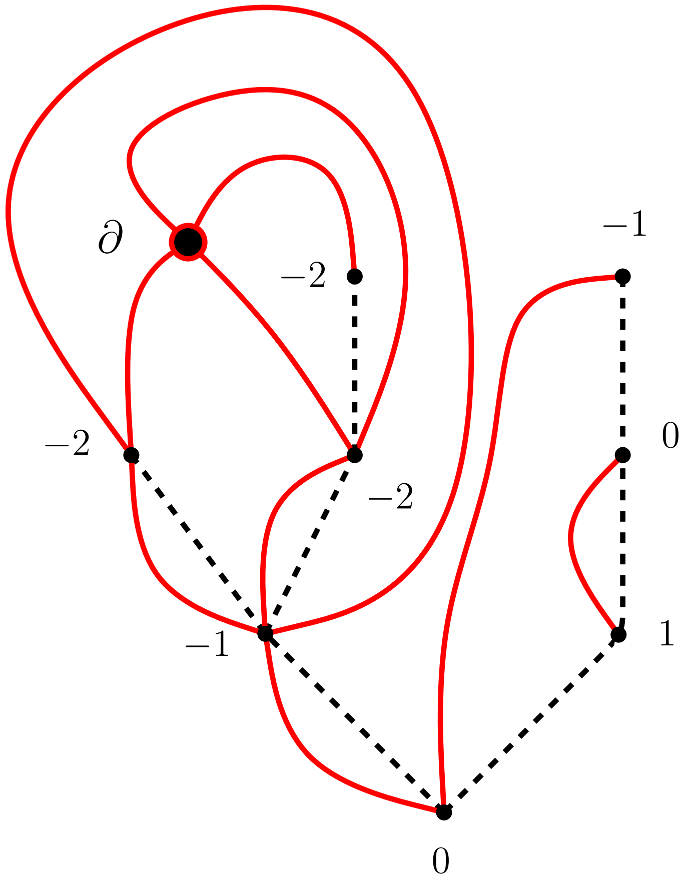
Schaeffer-type constructions



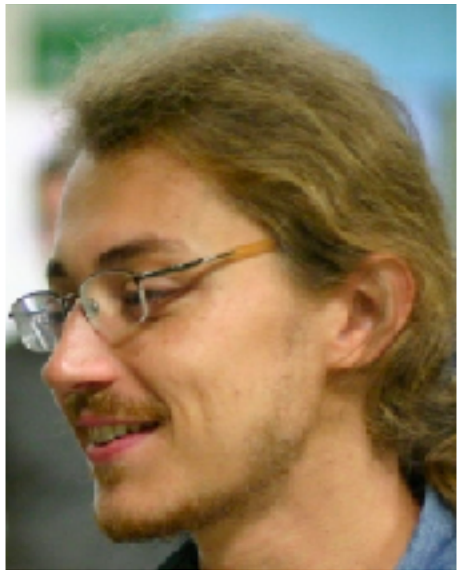


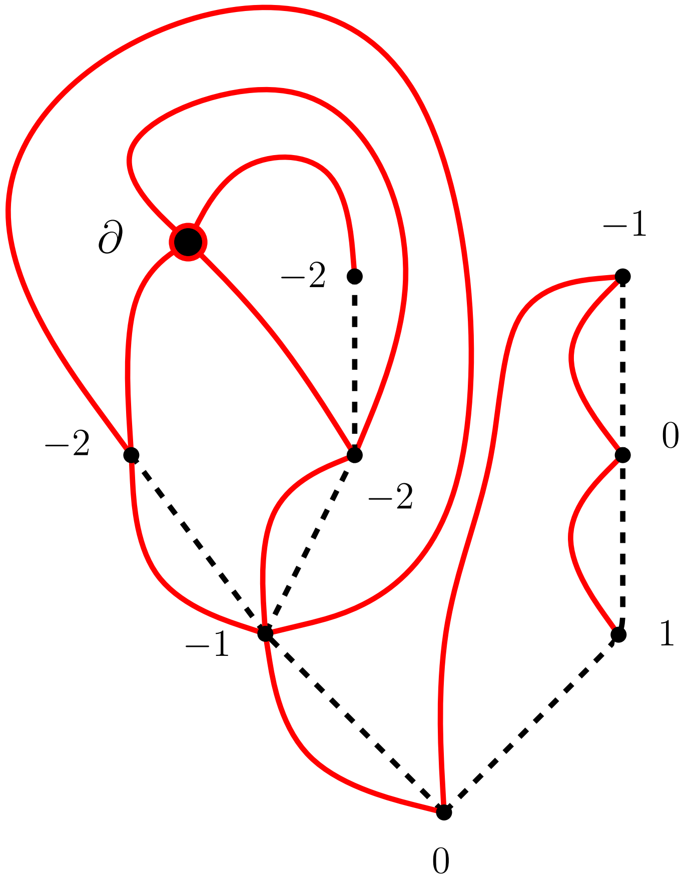
Schaeffer-type constructions



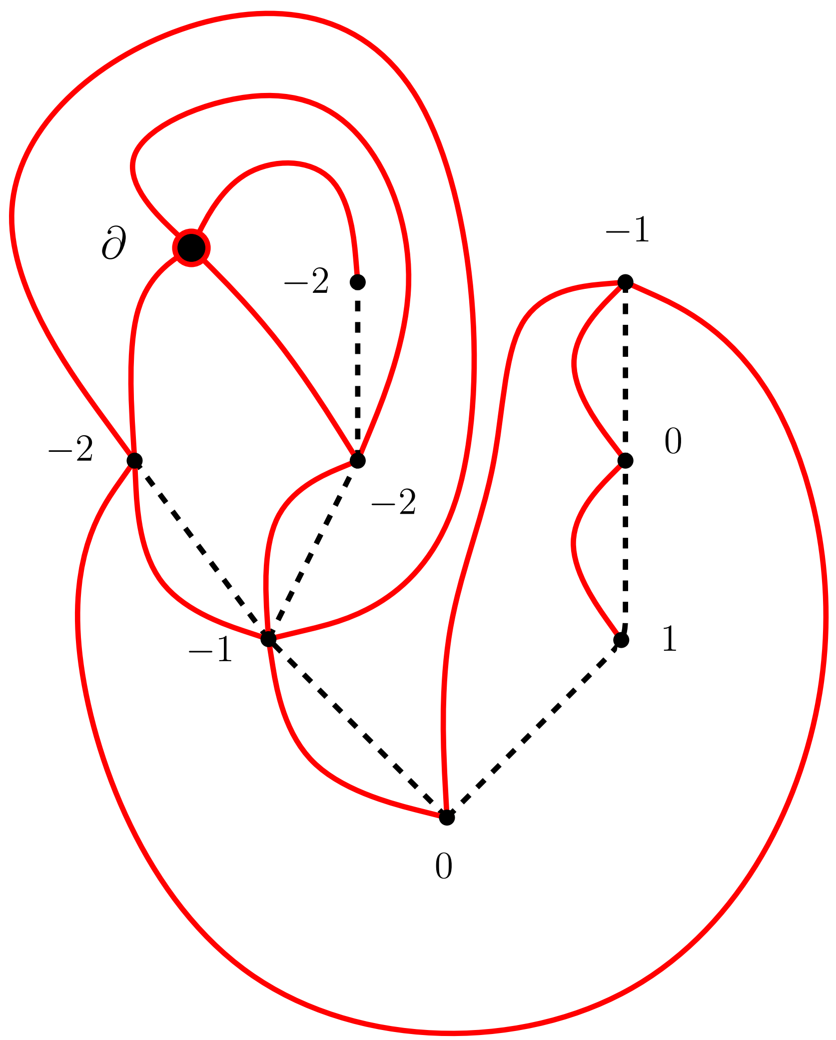


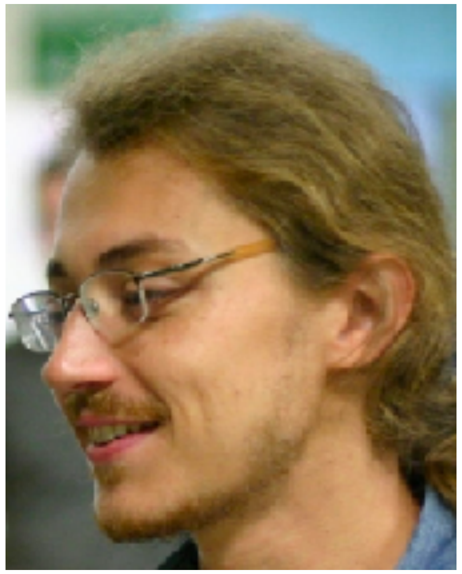
Schaeffer-type constructions





Schaeffer-type constructions





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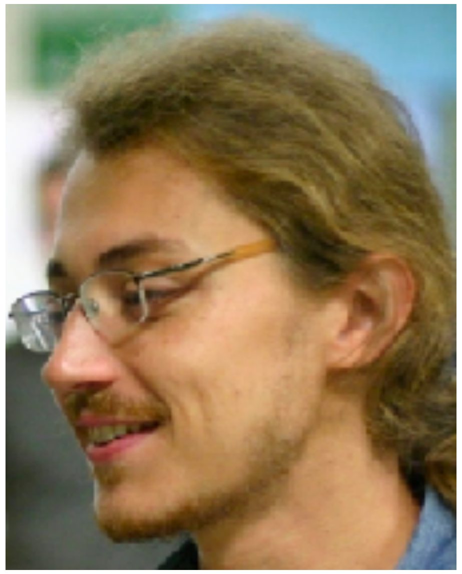
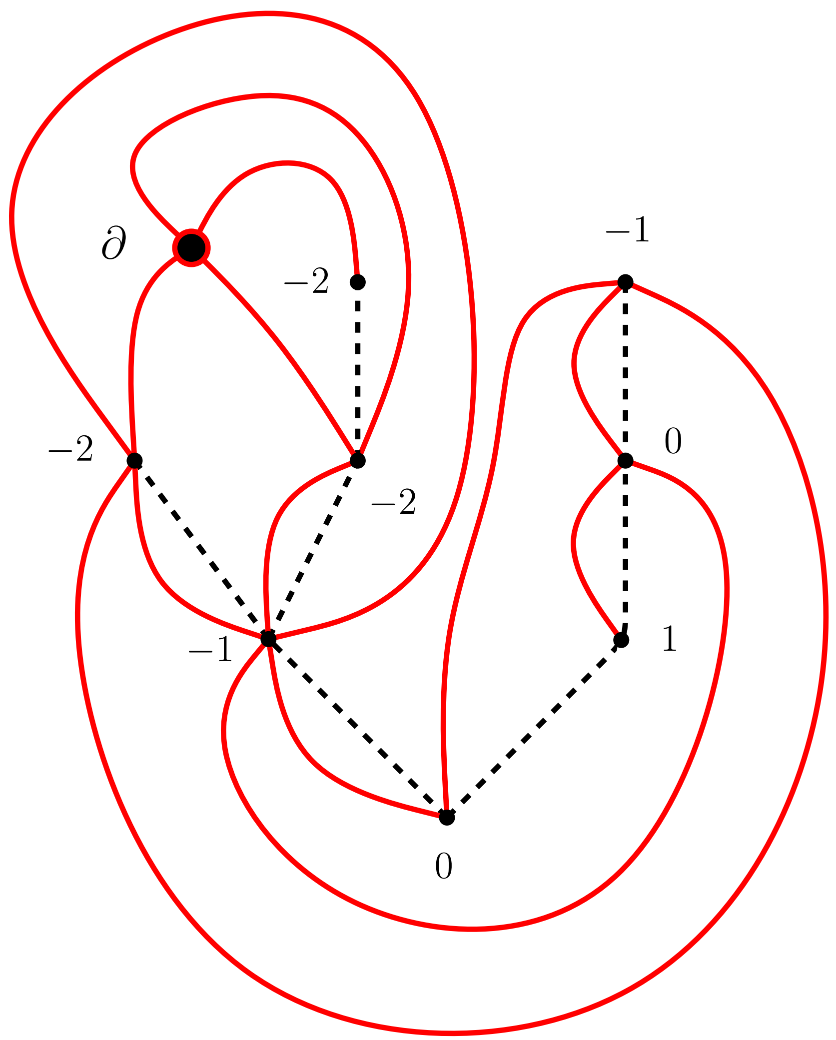
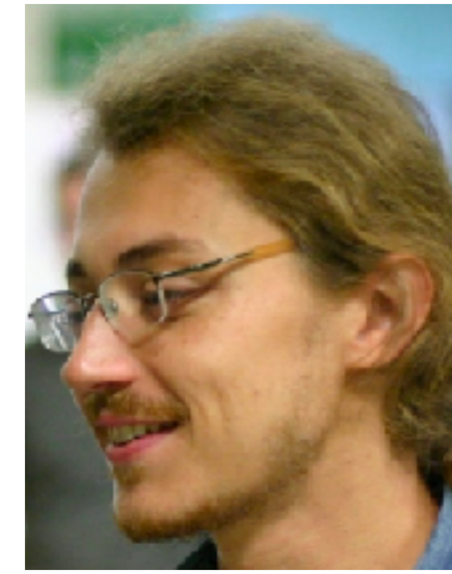
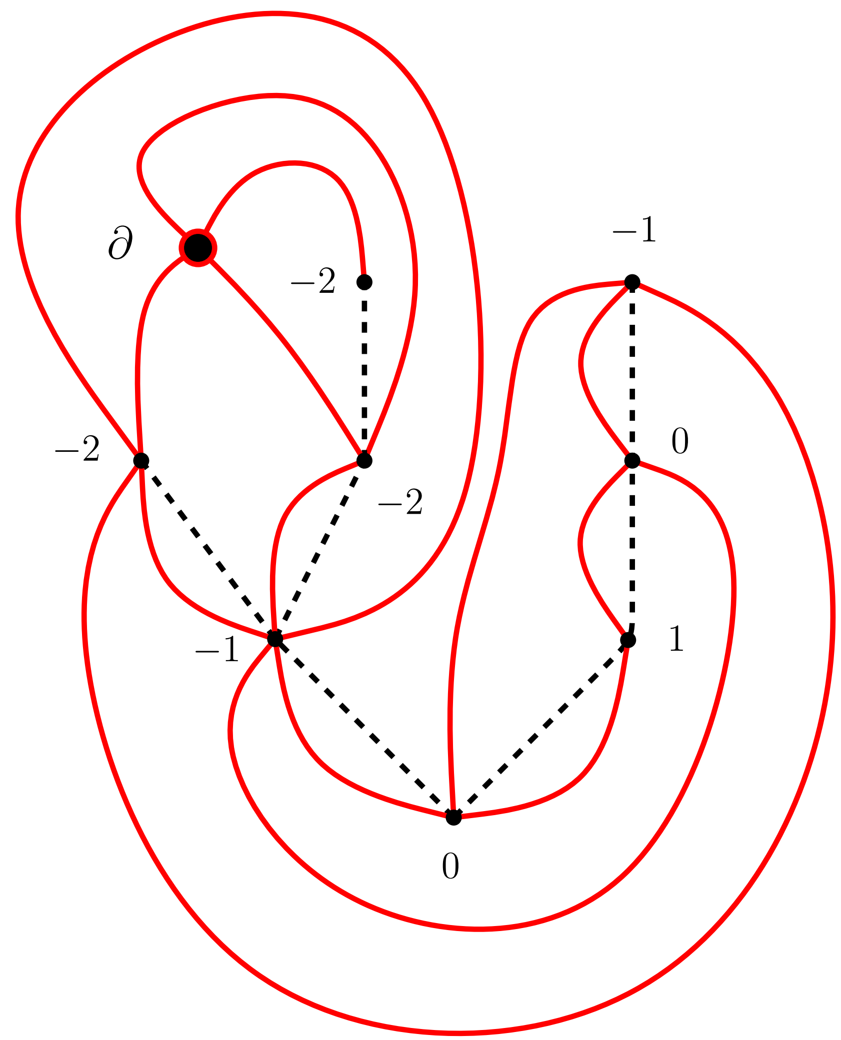


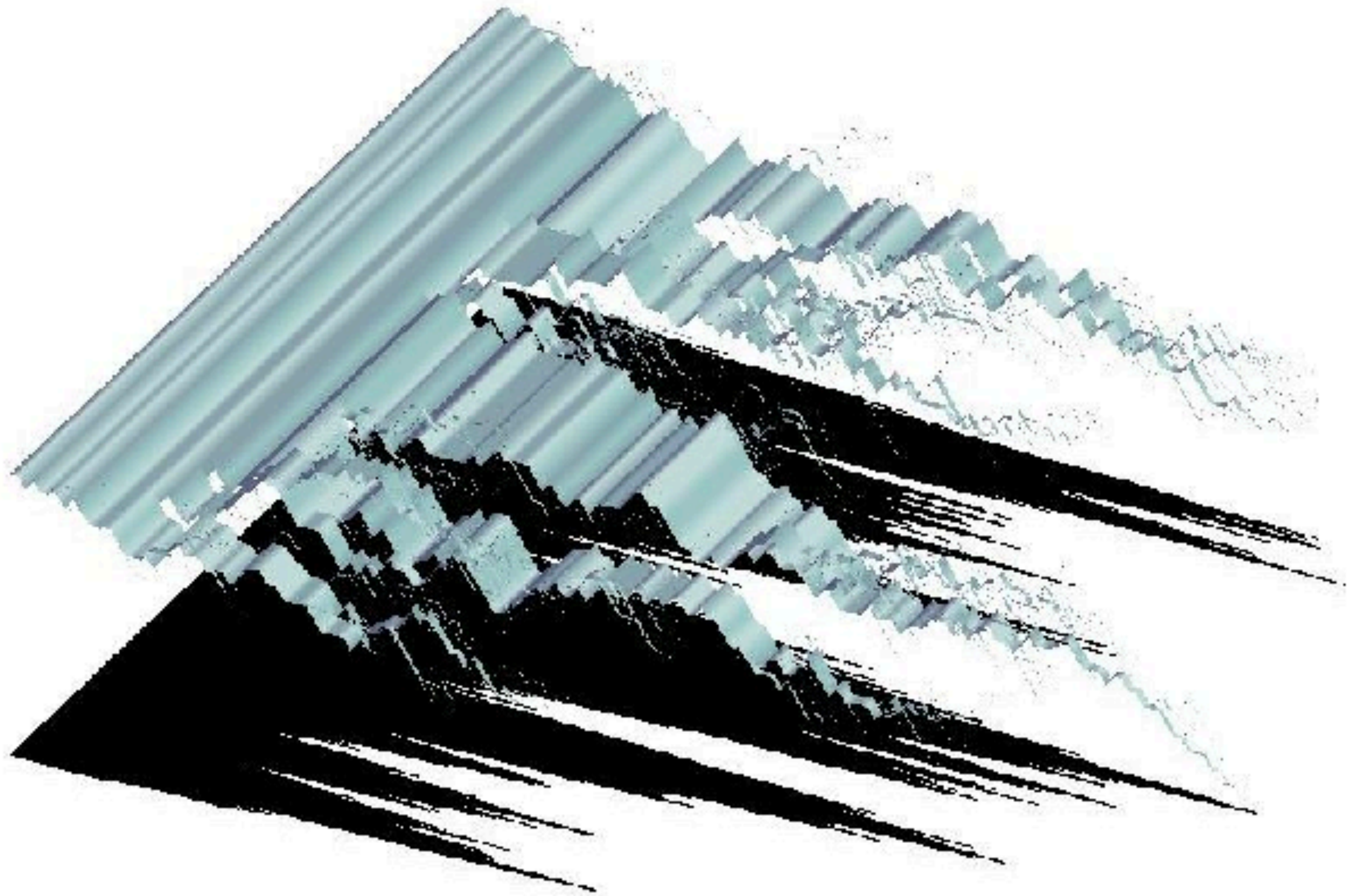


Schaeffer-type constructions





Schaeffer-type constructions

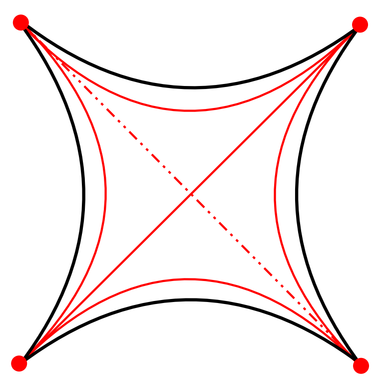


Chassaing-Schaeffer
Marckert-Mokkadem
Miermont
Le Gall
....

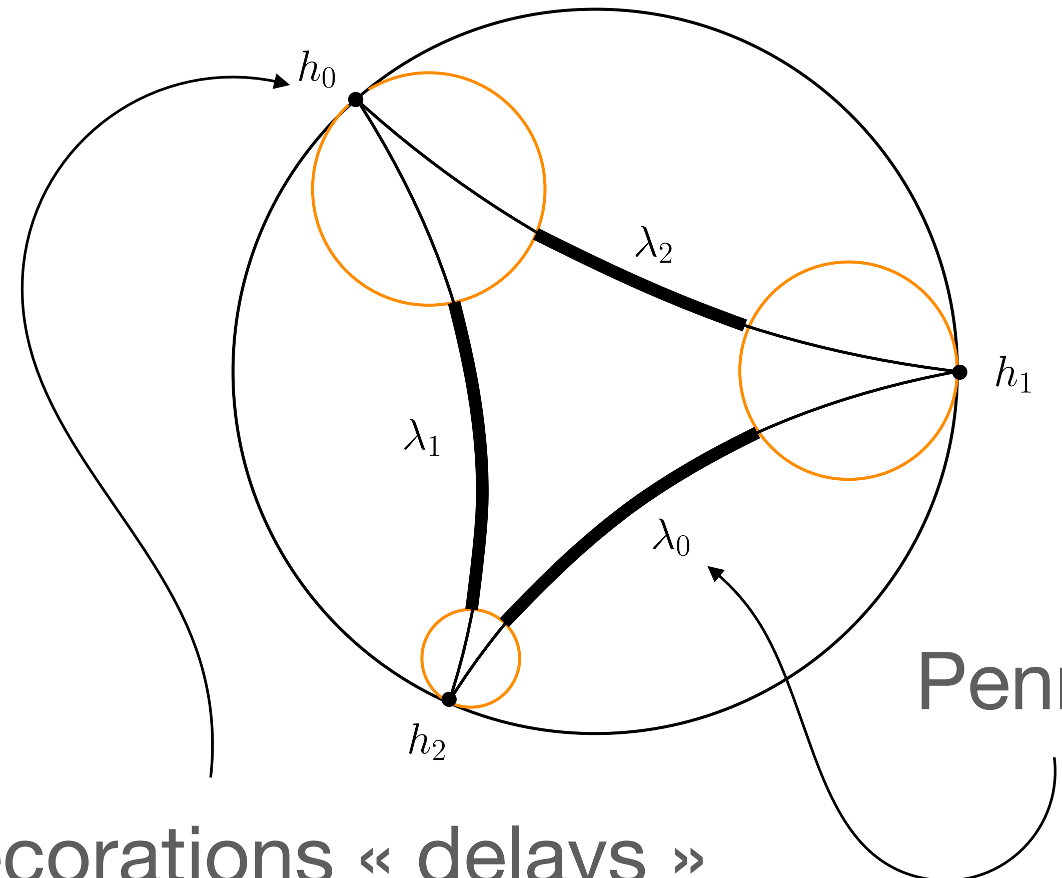
Courtesy of J. Bettinelli

Scaling limits of contour and label process of random labeled tree -> Le Gall’s Brownian snake
Main ingredient in the proof of the convergence towards the Brownian sphere

Penner's decorated Teichmüller theory



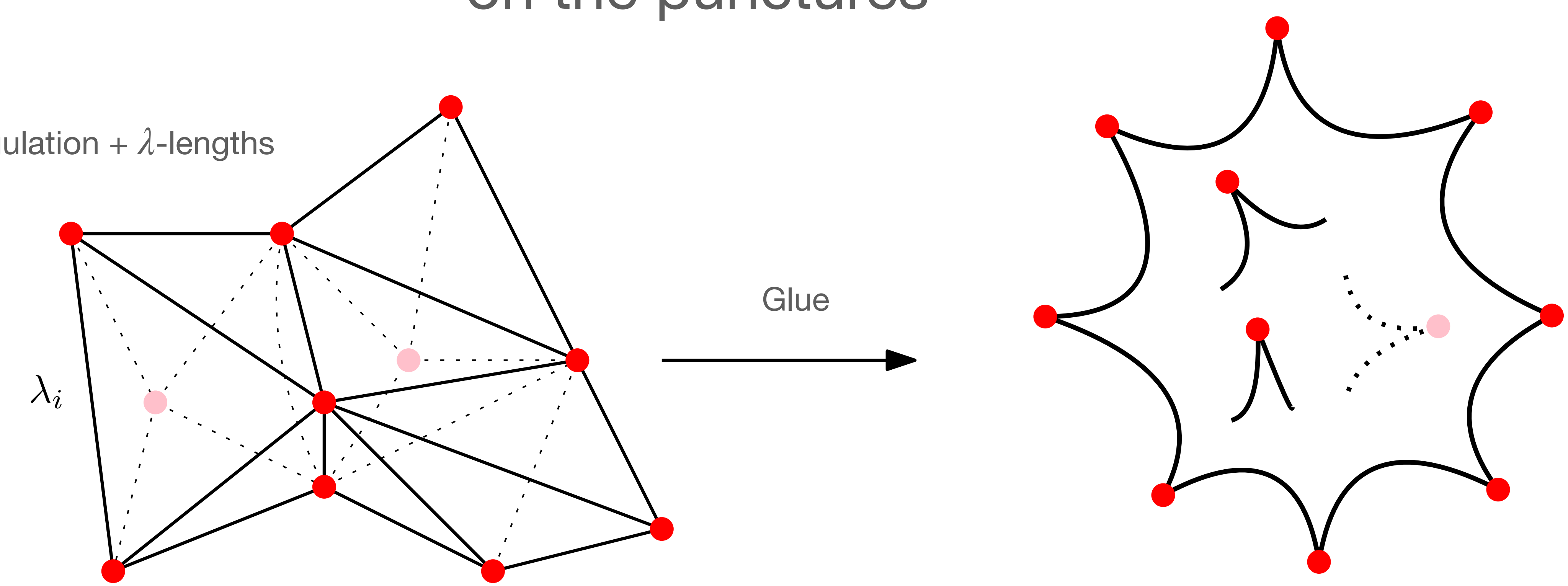
Triangulation
(Vertices at cusps)



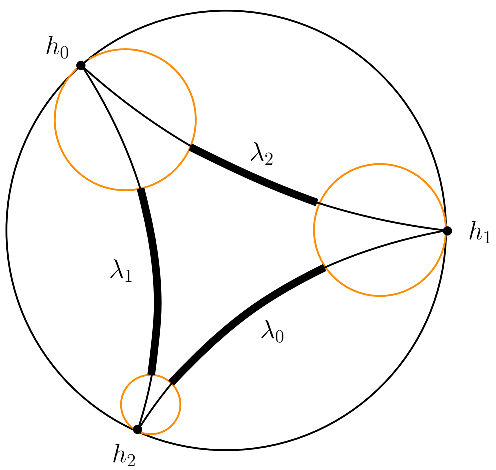
Penner's lambda-lengths
(Can be negative)

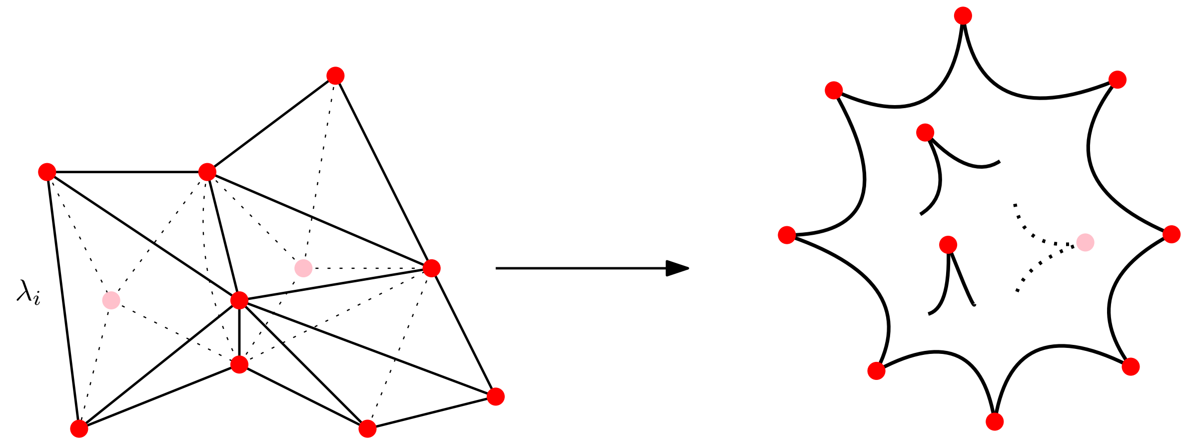
Decorations « delays »
on the punctures

Triangulation + λ-lengths



Penner’s decorated Teichmüller theory





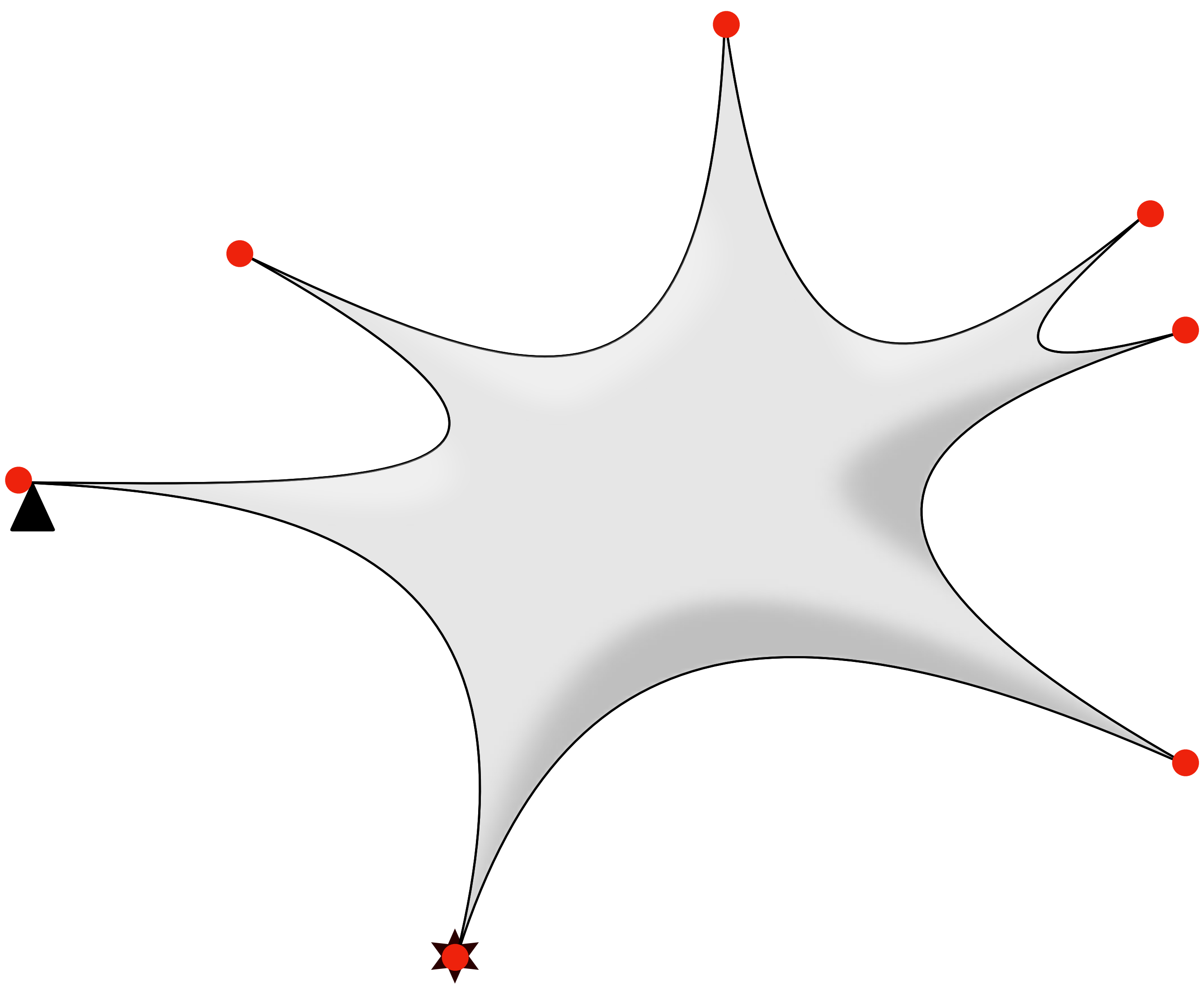
Weil-Peterson measure is expressed nicely in this coordinate system.

Here also, this is over-parametrized. As before, if we impose some geometric condition on the labeled triangulation on can specify uniquely the coordinates

-> Bowditch-Epstein-Penner Voronoï construction

Bowditch-Epstein-Penner Voronoï construction

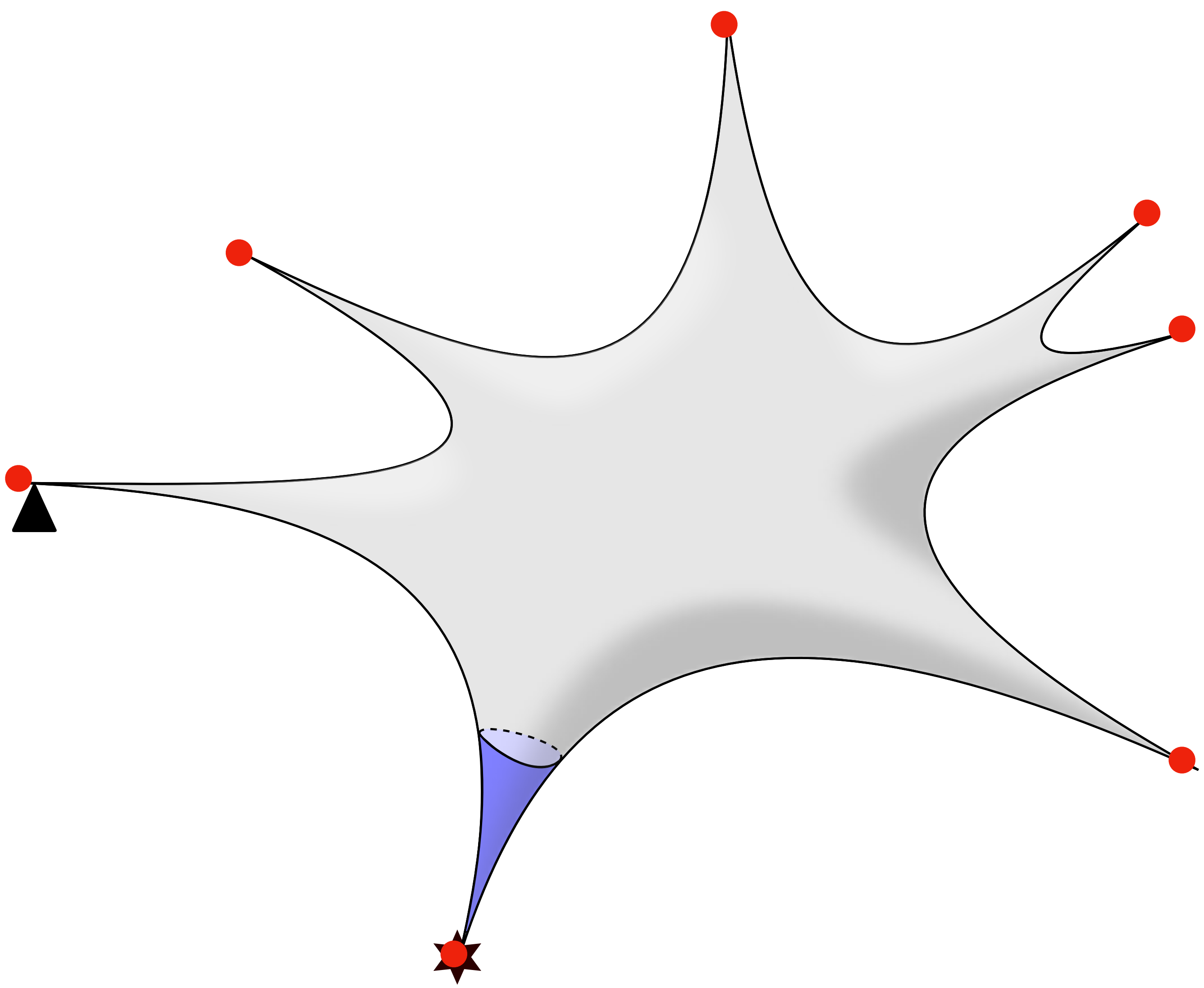
$g = 0$ with n punctures



Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

$g = 0$ with n punctures



Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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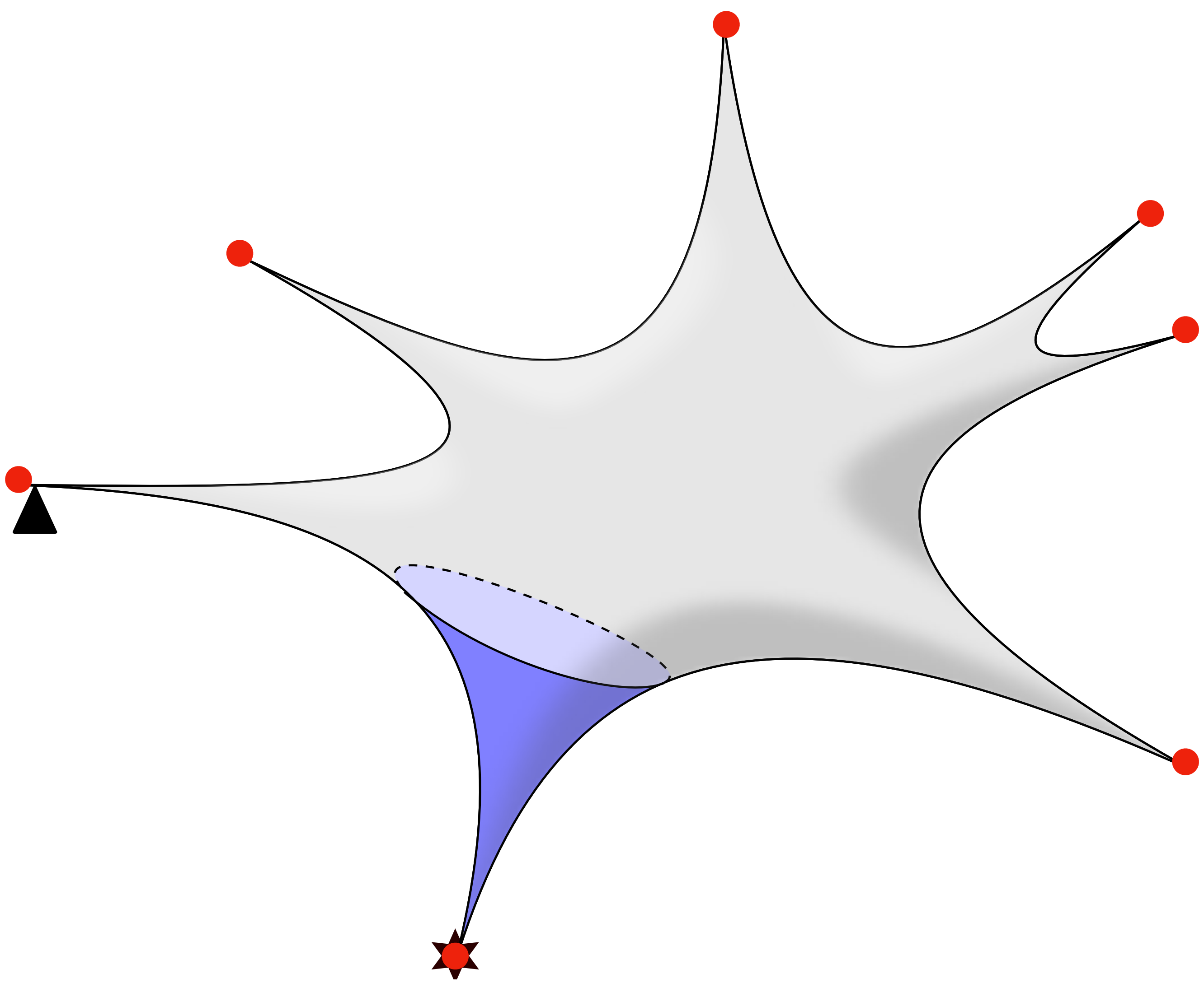

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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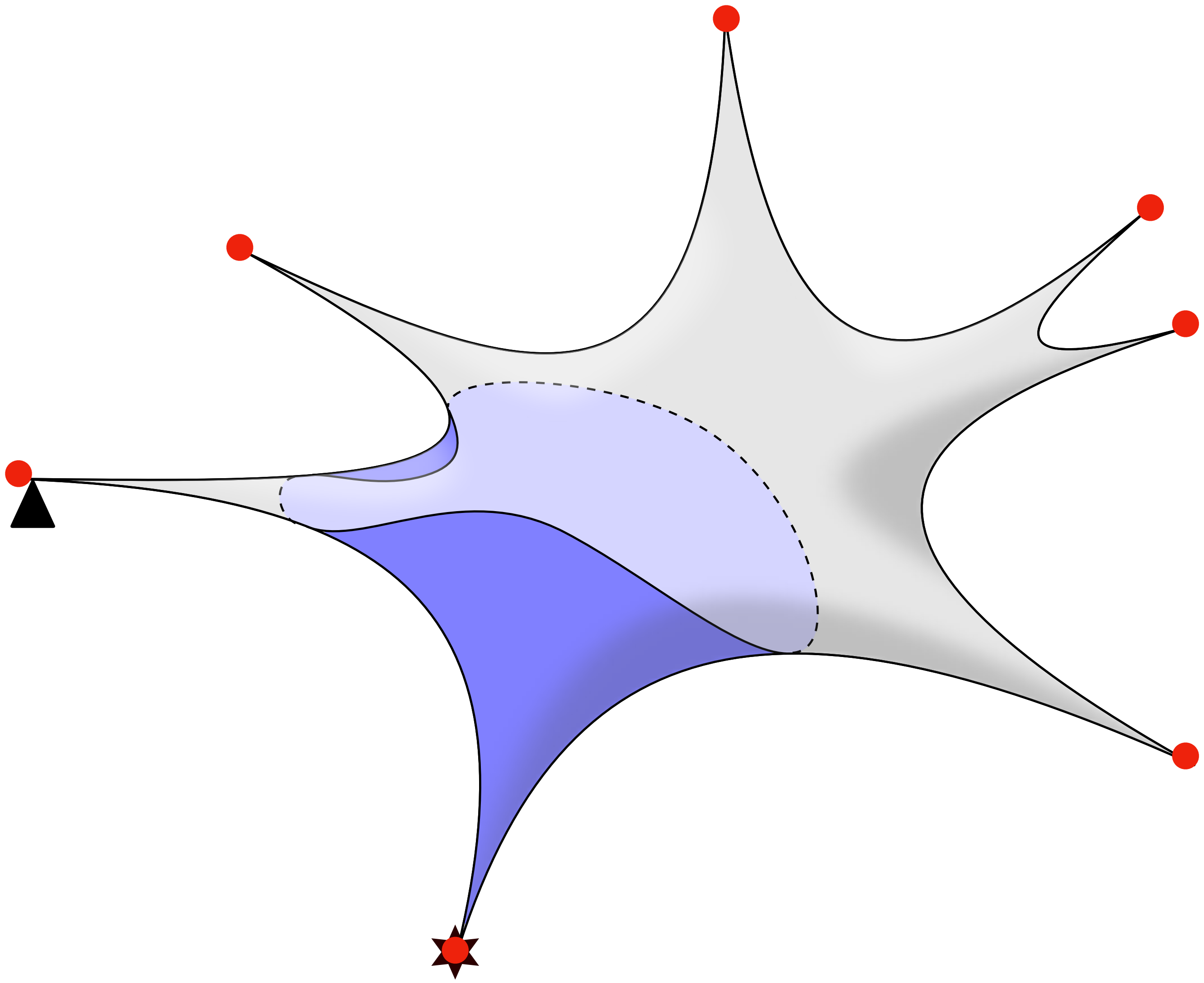

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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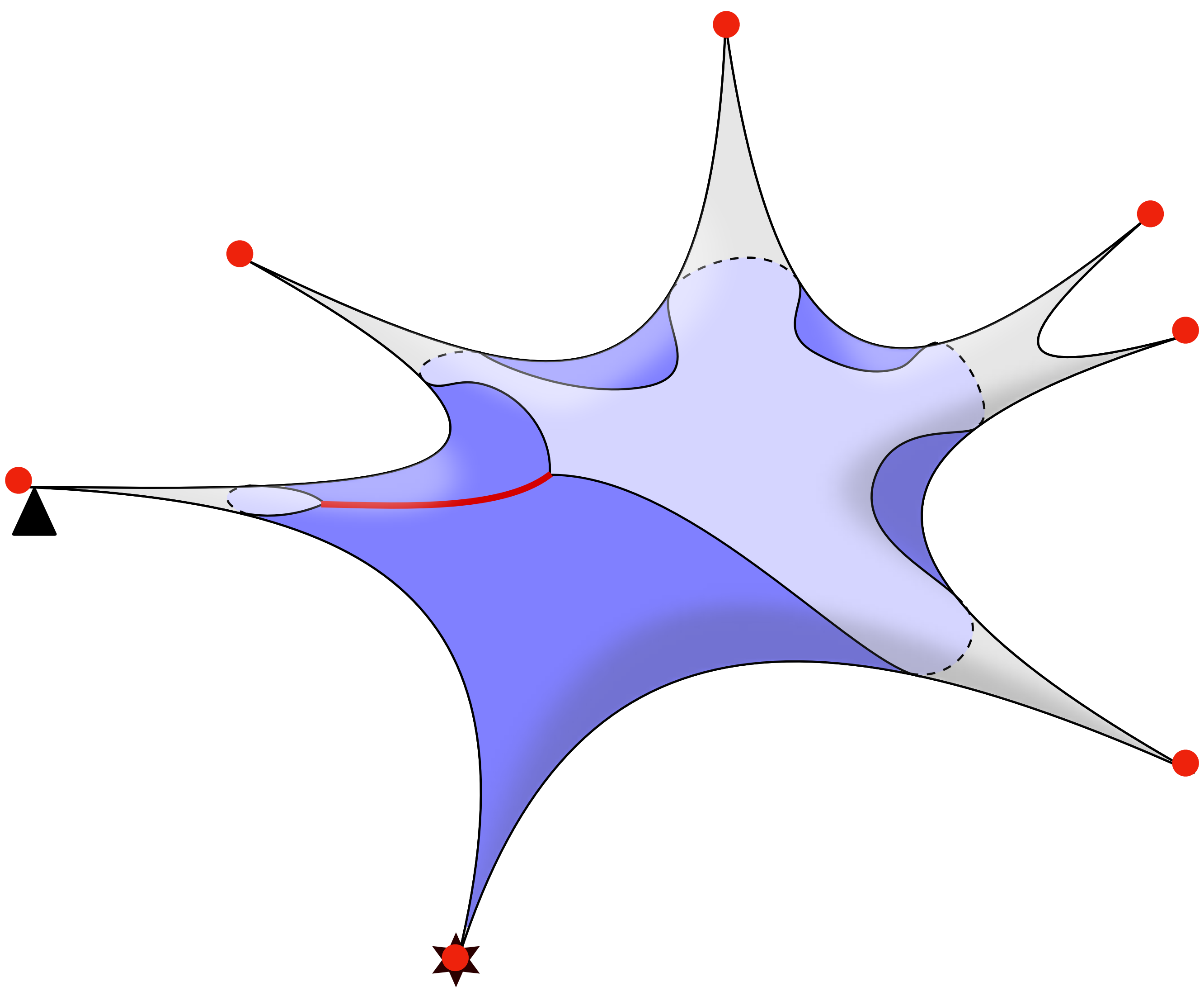

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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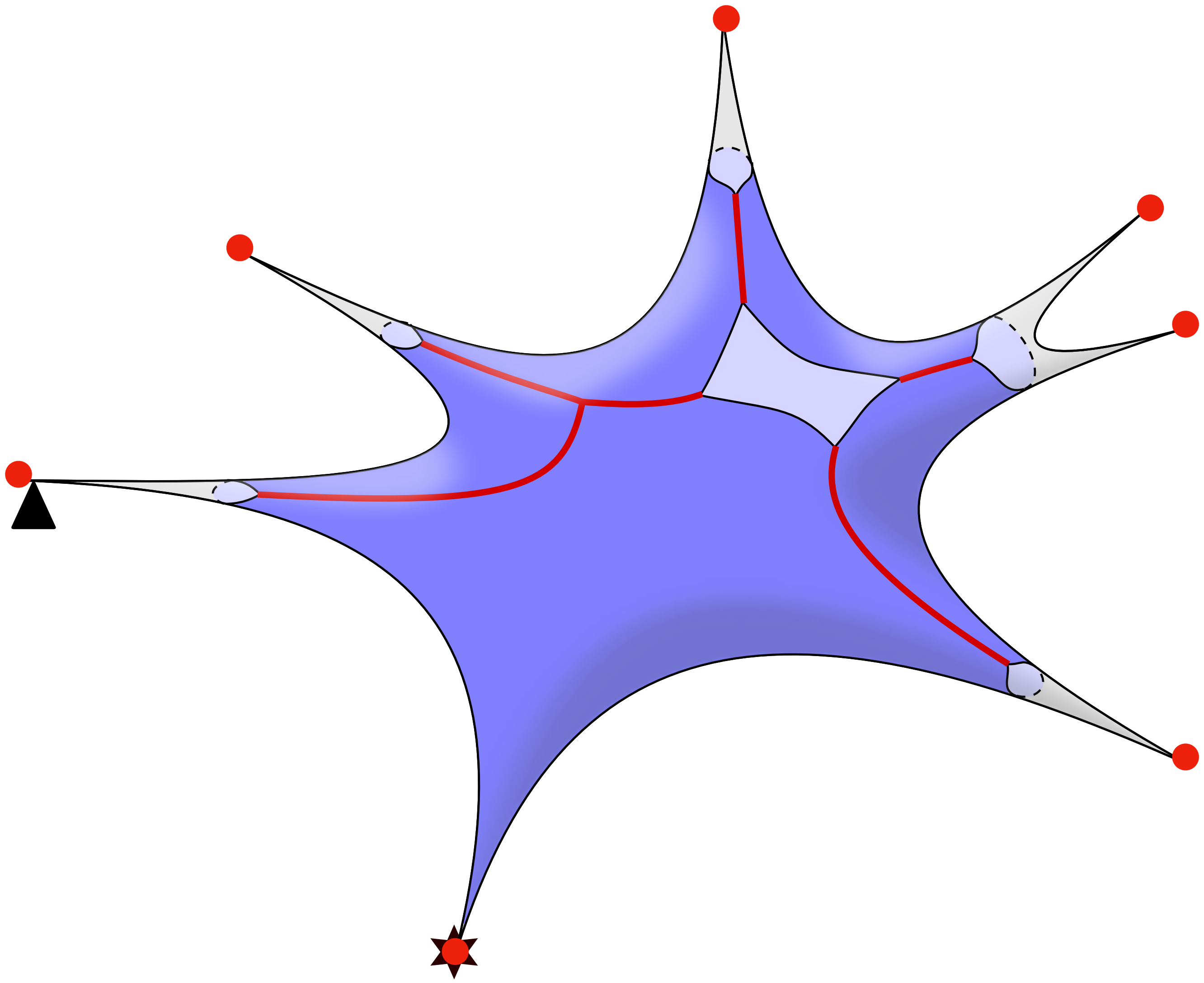

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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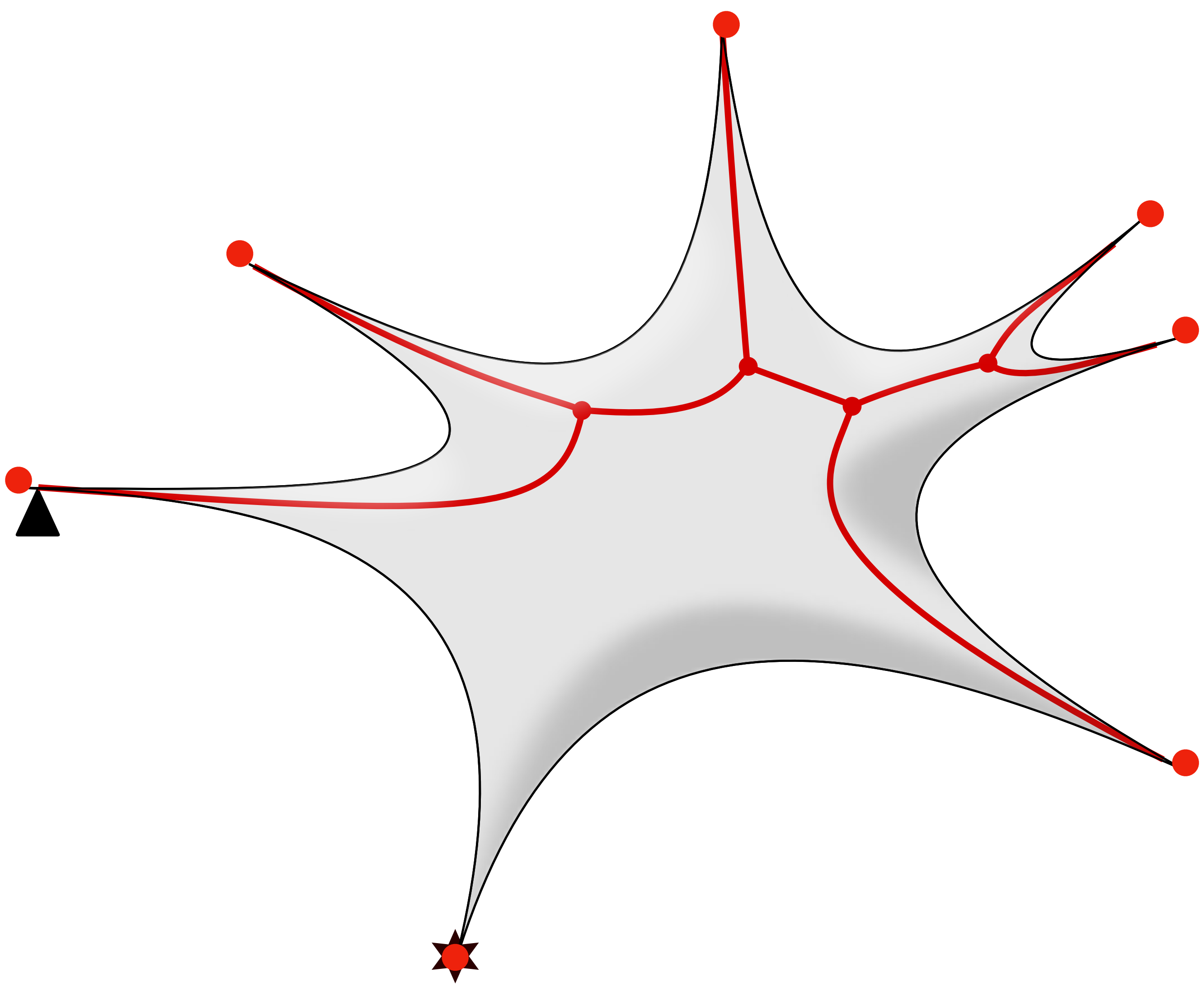

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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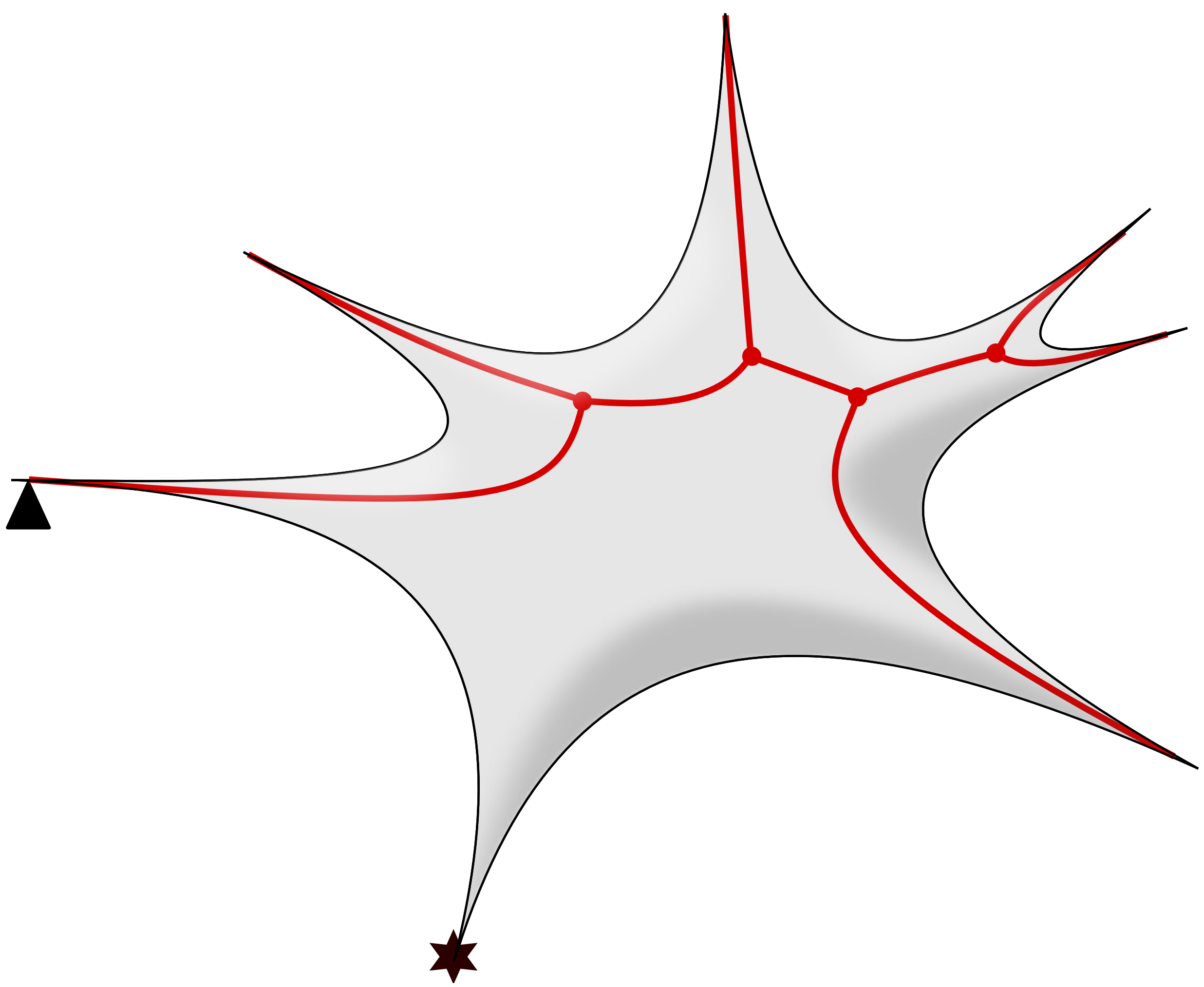
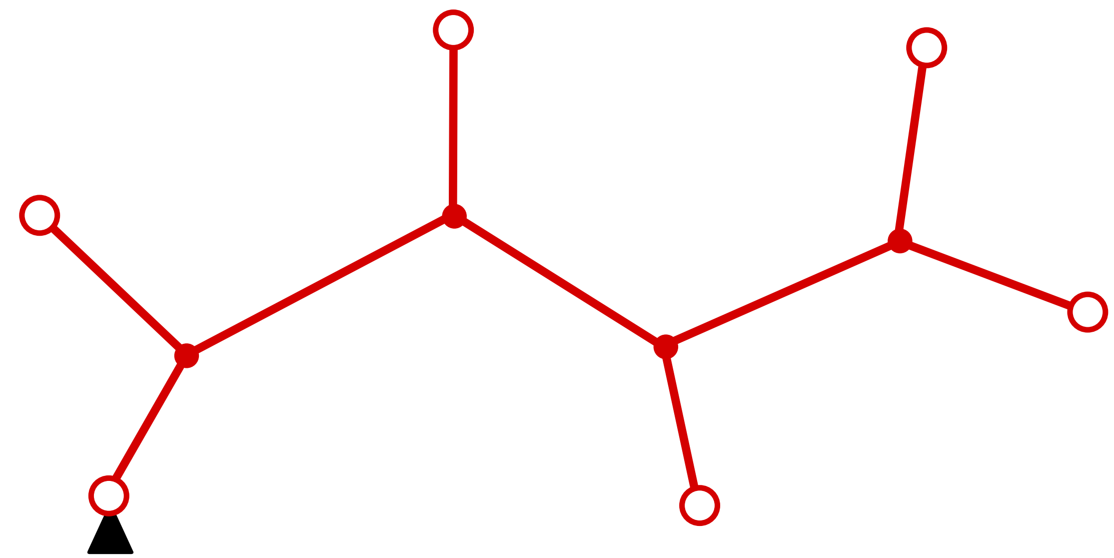


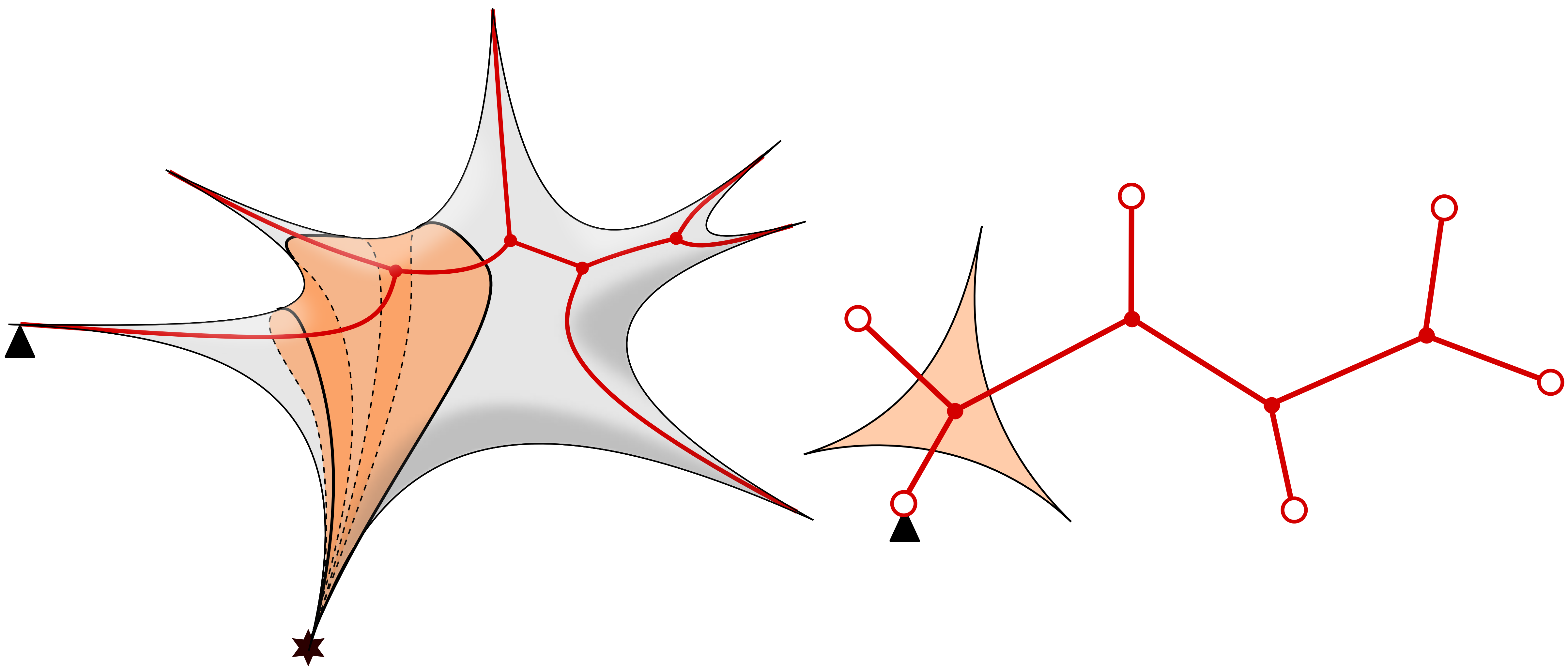


spine/cutlocus

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

$g = 0$ with n punctures



Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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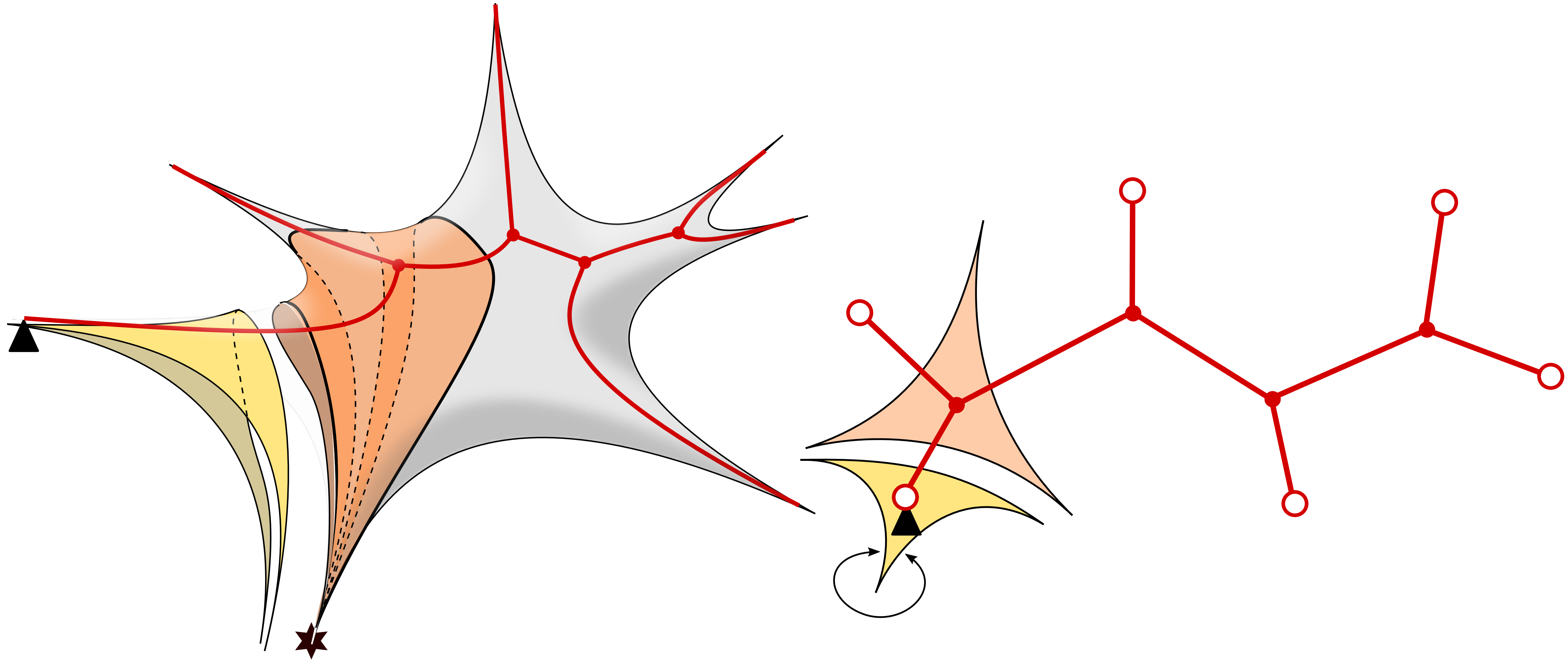

Courtesy of T. Budd

Bowditch-Epstein-Penner Voronoï construction

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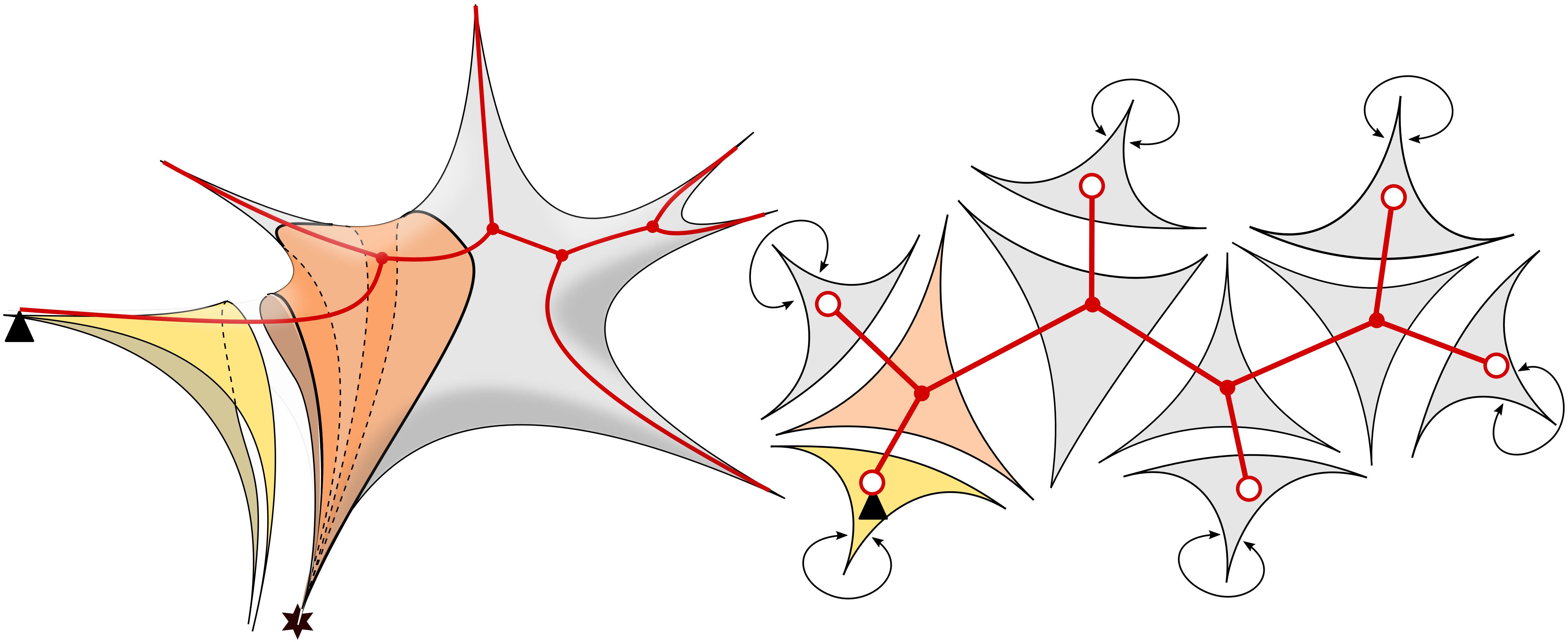


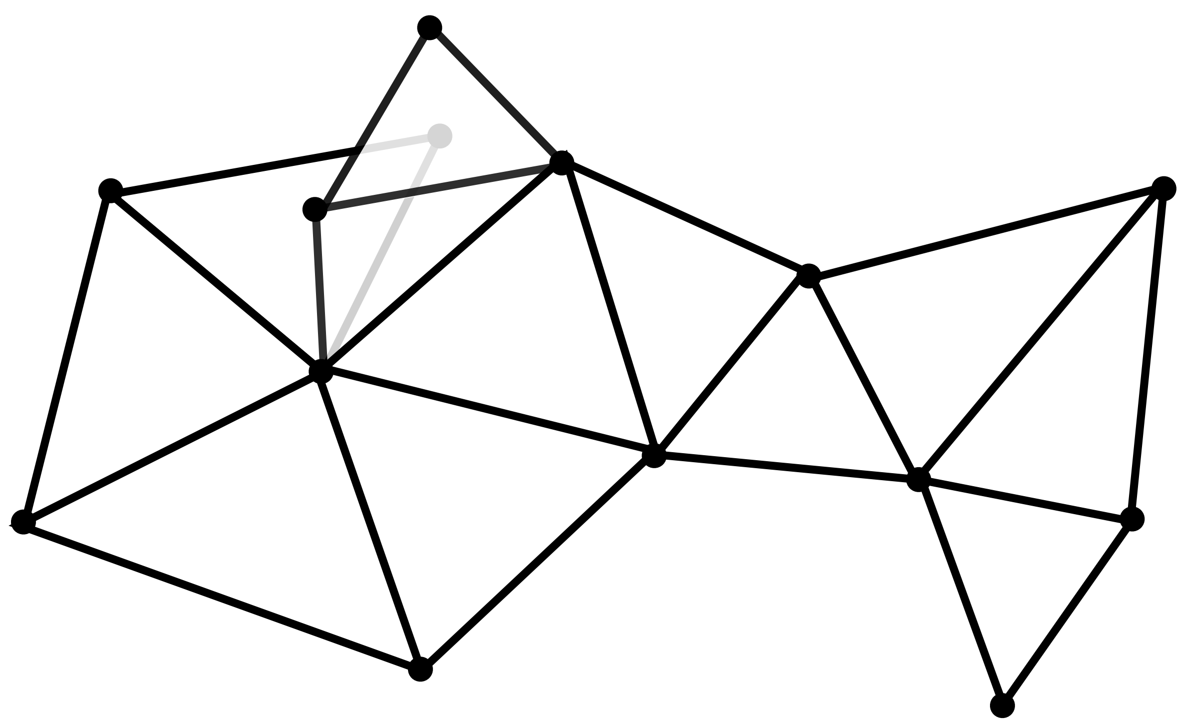
Courtesy of T. Budd

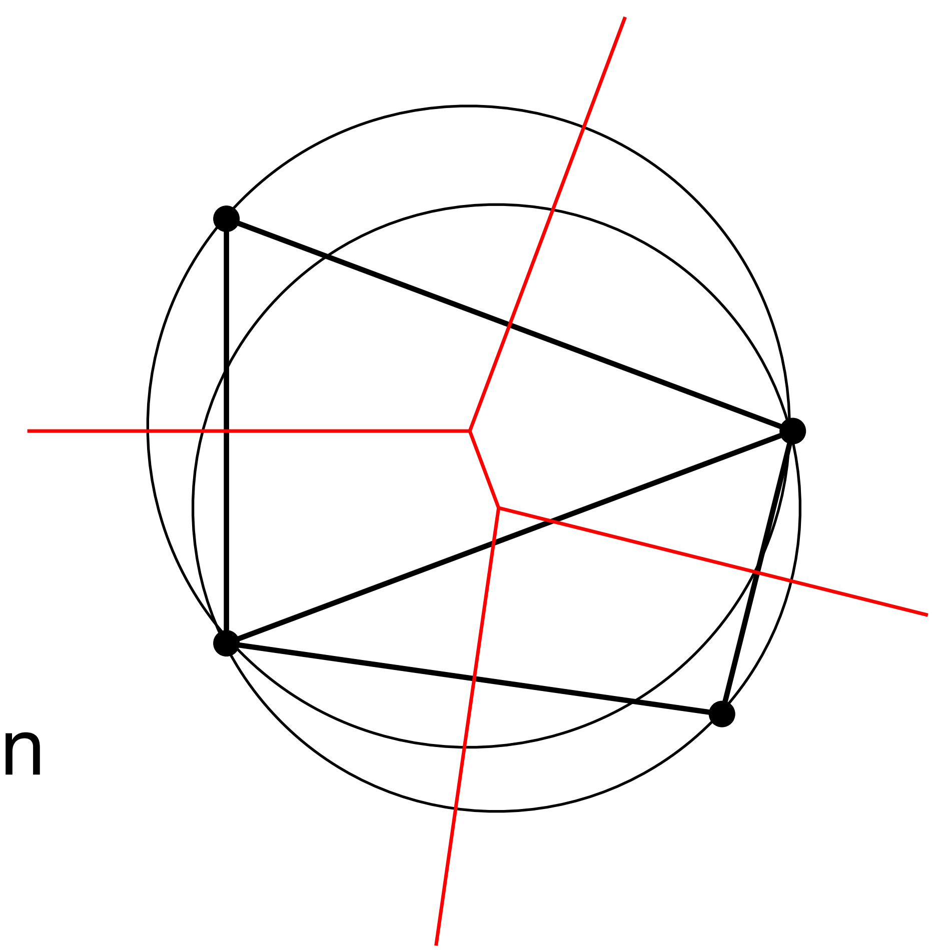
Bowditch-Epstein-Penner Voronoï construction

$g = 0$ with n punctures

In Penner's λ-lengths, represented as Euclidean triangles

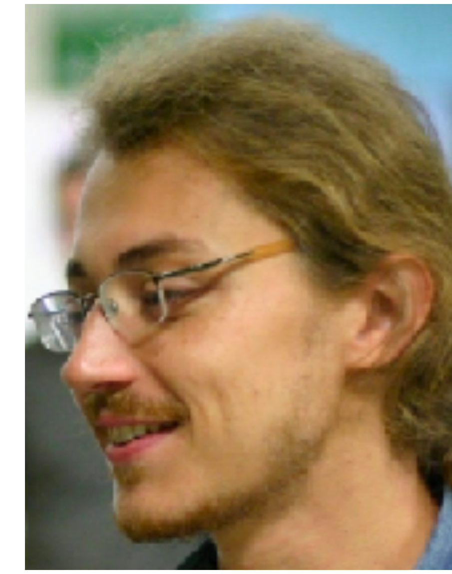
$$\ell_e = \mathrm{e}^{\lambda_e/2}$$



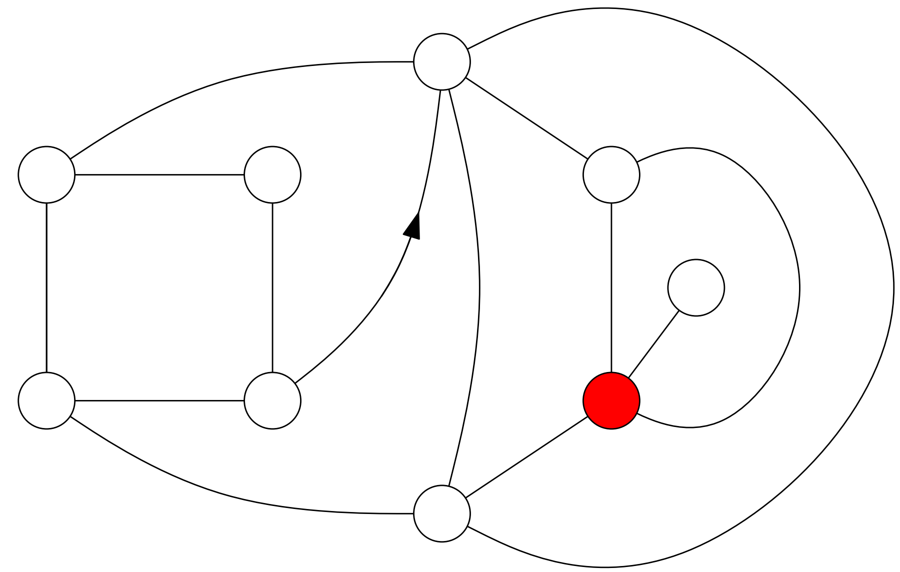


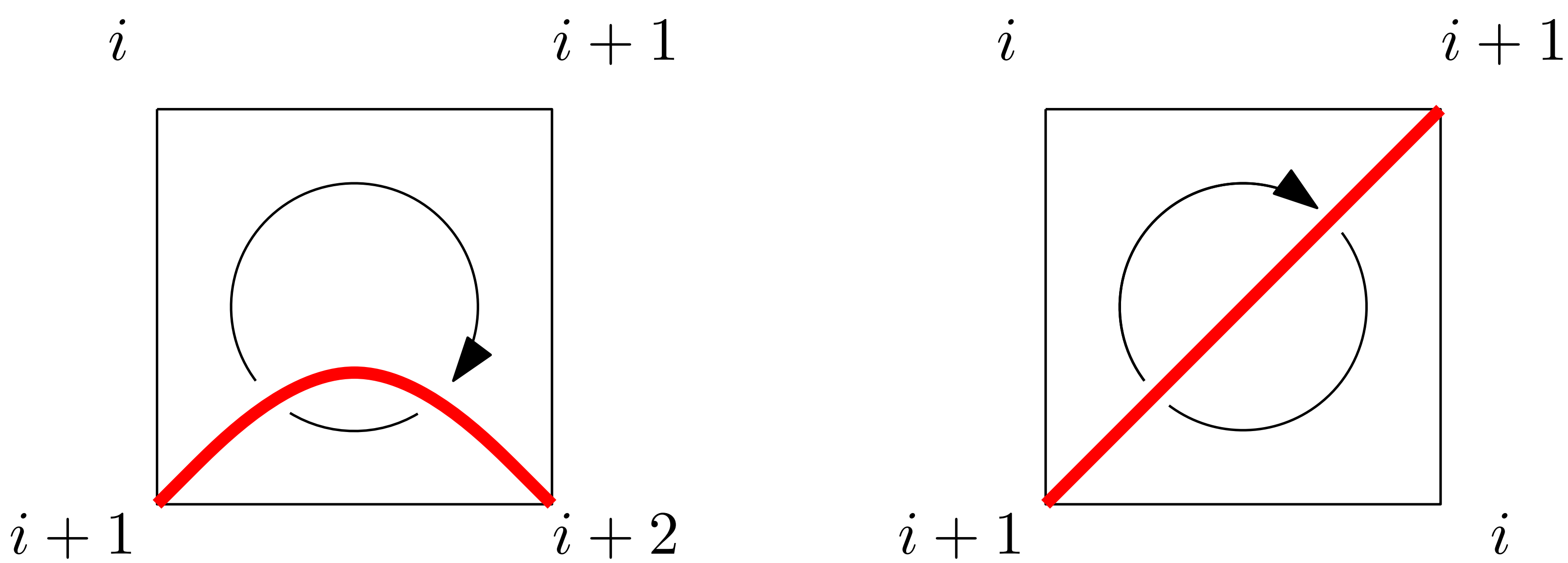
Voronoï condition

Back to Schaeffer

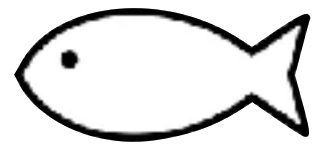


Quadrangulation





There’s something fishy about it, isn’it ?



LET NONE BUT GEOMETERS ENTER HERE

Thanks !