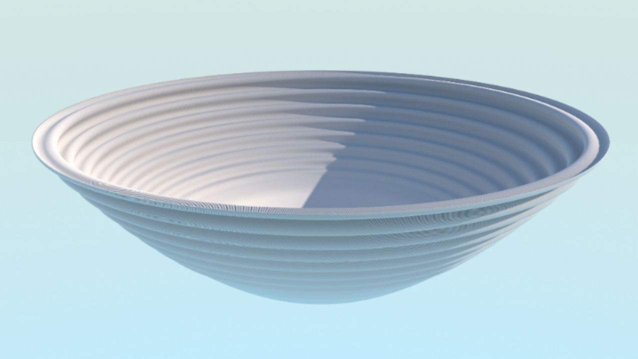
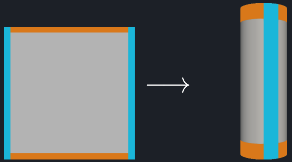
A C^1-isometric embedding of the hyperbolic plane and its limit set



Mélanie Theillière, University of Luxembourg
joint work with V. Borrelli, R. Denis, F. Lazarus, B. Thibert

Nash-Kuiper Theorem

An isometric map



Lengths are preserved by the map.

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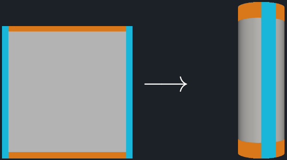

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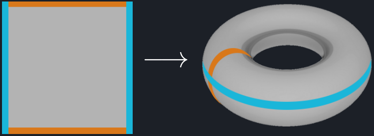
A non-isometric map



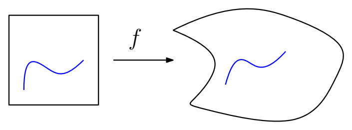
At the left, the blue and the orange curves have the same length. This is no longer the case at the right.

Nash-Kuiper Theorem

Definition

Let $f : ([0,1]^m, g) \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, with g a metric. The map f is isometric if, for any curve γ in $[0,1]^m$, we have

$$length_g(\gamma) = length_{\langle\cdot,\cdot\rangle}(f \circ \gamma)$$



or, equivalently, if $g = f^*\langle\cdot,\cdot\rangle$.

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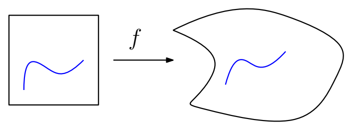


or, equivalently, if $g = f^* \langle \cdot, \cdot \rangle$.

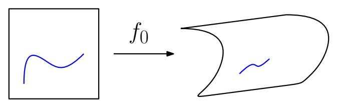
Theorem (Hilbert, 1901 - Efimov, 1964)

There is no C^2-isometric immersion of $\mathbb{H}^2$ in $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$.

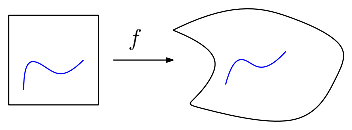
Nash-Kuiper Theorem

Nash-Kuiper C^1-isometric embedding Theorem, 1954-55

If there exists $f_0 : ([0,1]^m, g) \to (\mathbb{R}^{m+1}, \langle \cdot, \cdot \rangle)$ a C^∞ strictly short embedding, ie $g - f_0^* \langle \cdot, \cdot \rangle > 0$,



then there exists C^1-isometric embeddings $f : ([0,1]^m, g) \to (\mathbb{R}^{m+1}, \langle \cdot, \cdot \rangle)$.



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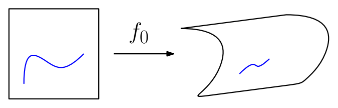

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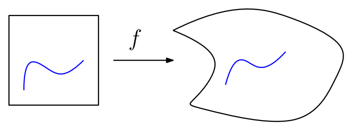


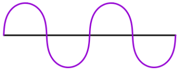
Corollary (Kuiper, 1955)

There exists C^1-isometric embeddings of $\mathbb{H}^2$ in $(\mathbb{R}^3, \langle \cdot, \cdot \rangle)$.

Idea of the construction

Main tool: a corrugation formula

$$f_1(x) := f_0(x) + \frac{1}{k} r \left[\begin{array}{c} \int_0^{Nx_j} \cos(a\cos(2\pi u)) - J_0(a) du \, \mathbf{T}(x) \\ + \int_0^{Nx_j} \sin(a \cos(2\pi u)) du \, \mathbf{N}(x) \end{array} \right]$$



with $\mathbf{T}$ a unit tangent vector, $\mathbf{N}$ a unit normal vector, $r, a \in \mathbb{R}$, $k \in \mathbb{N}$, and $J_0(a) = \int_0^1 \cos(a\cos(2\pi u))du$.

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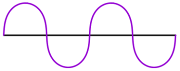

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Properties

If for any $x \in [0,1]^m$ we have $\partial_j f_0(x) = rJ_0(a)\mathbf{T}(x)$, then:

- $\|f_1 - f_0\|_{C^0} = O(1/k)$
- $\|\partial_j f_1\|_{\mathbb{R}^n} = r + O(1/k)$
- $\|\partial_i f_1 - \partial_i f_0\|_{C^0} = O(1/k)$, for any $i \neq j$

Idea of the construction

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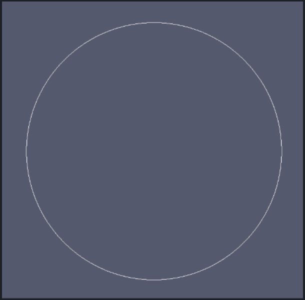
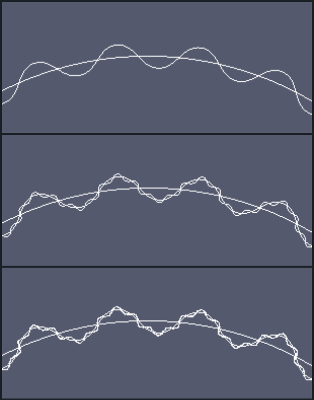


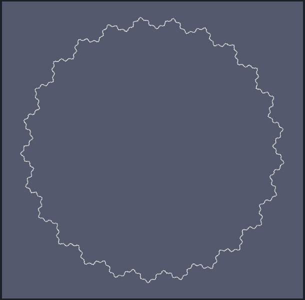
$\|f_0'\|^2 \equiv 1/2$

- Iteration -



$\|f_1'\|^2 \equiv 3/4,$
$\|f_2'\|^2 \equiv 7/8, \ldots$

- Output -

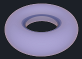


at the tlimit, $\|f_\infty'\|^2 \equiv 1$

Idea of the construction

Strategy for a surface (torus case)

Input. Let $f_0 : ([0,1]^2, g) \to (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$, with g the usual euclidean metric, be a short map.



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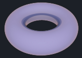

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$$\Delta \quad := \quad g - f_0^* \langle \cdot, \cdot \rangle > 0$$

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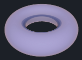

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$$\begin{aligned} \Delta &:= g - f_0^*\langle\cdot,\cdot\rangle > 0 \\ &= \left(\mathrm{d}x^2 + \mathrm{d}y^2\right) - \left(E\mathrm{d}x^2 + 2F\mathrm{d}x\mathrm{d}y + G\mathrm{d}y^2\right) \end{aligned}$$

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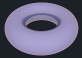

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We decompose Δ as a sum of squares of linear forms:

$$\Delta = \rho_1 \mathrm{d}x^2 + \rho_2 \left(\frac{1}{\sqrt{5}}(\mathrm{d}x + 2\mathrm{d}y)\right)^2 + \rho_3 \left(\frac{1}{\sqrt{5}}(\mathrm{d}x - 2\mathrm{d}y)\right)^2$$

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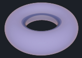

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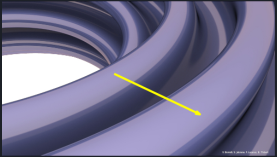
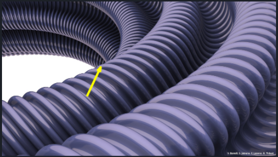
-> Note that the choice of directions is not unique.

Idea of the construction

Main Step. We reduce by 2 the isometric default adding corrugations in three directions



∂_x



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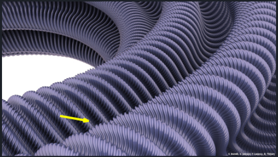

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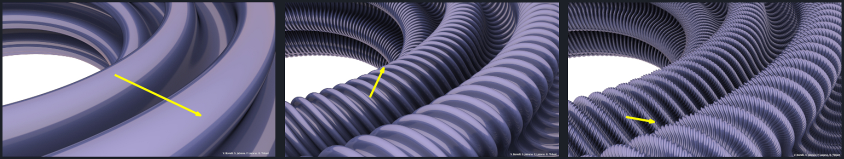

∂_x $\qquad\qquad$ $\frac{1}{\sqrt{5}}(\partial_x + 2\partial_y)$ $\qquad\qquad$ $\frac{1}{\sqrt{5}}(\partial_x - 2\partial_y)$

Output. At the limit, we have

and at the limit the isometric default

$$\frac{1}{2^j}\Delta \quad \underset{j\to+\infty}{\longrightarrow} \quad 0$$

Idea of the construction

Poincaré disk case

Let

$$f_0 : \quad (]0,1] \times [0, 2\pi[, h) \longrightarrow (\mathbb{R}^3, \langle \cdot, \cdot \rangle)$$
$$(\rho, \theta) \longmapsto 2(\rho \cos\theta, \rho \sin\theta, \tfrac{\sqrt{2}}{2}\rho^2)$$

where $h = \frac{4}{(1-\rho^2)^2}(d\rho^2 + \rho^2 d\theta^2)$ is the Poincaré metric.

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But $h \underset{\rho \to 1}{\not\to} \infty$ on the edge of the disk and $\Delta \underset{\rho \to 1}{\not\to} \infty$ too! **Fail!**

Idea of the construction

Next try. Let Δ_j be the truncated Taylor series of $\Delta = h - f_0^*\langle\cdot,\cdot\rangle$.

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At each step we increase the length until Δ_j (which is bounded).

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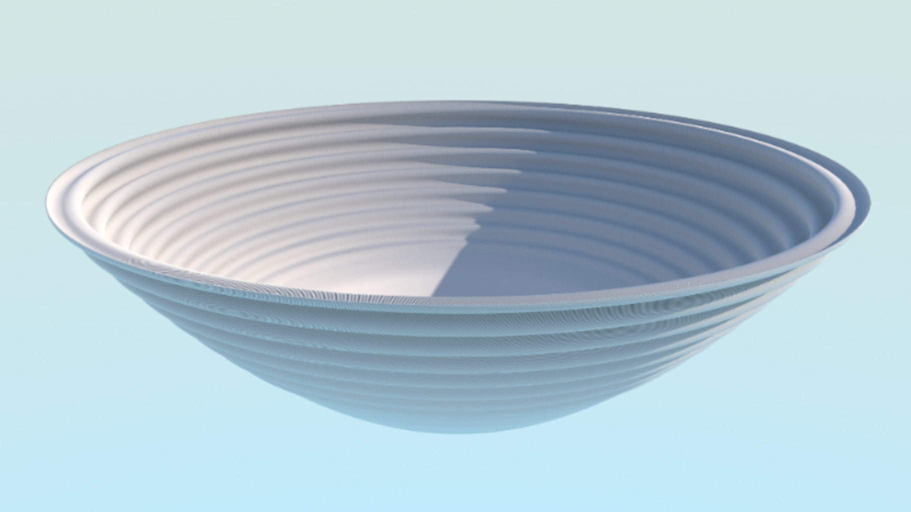
With these choices, we can prove the C^1-convergence so, **at the end**, we obtain a C^1-isometric embedding of $\mathbb{H}^2$.

Some pictures



(pictures made by Roland Denis of the Hevea team)

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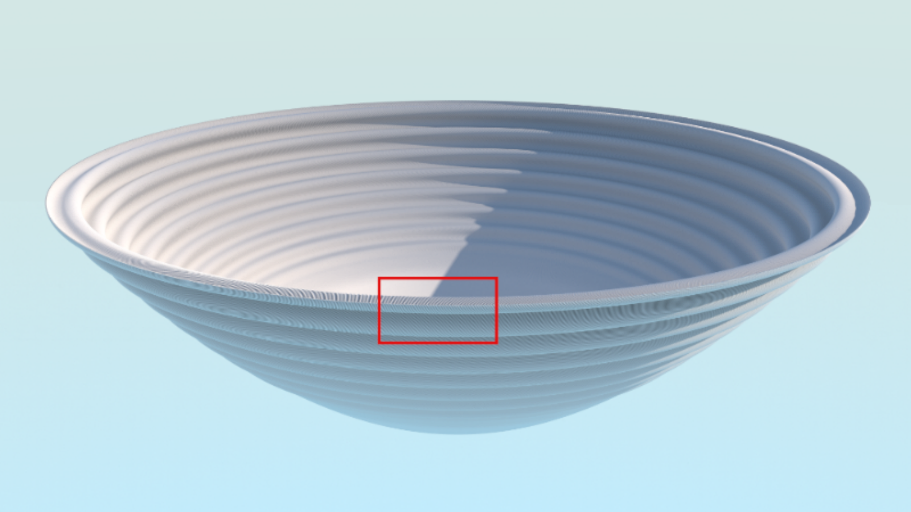

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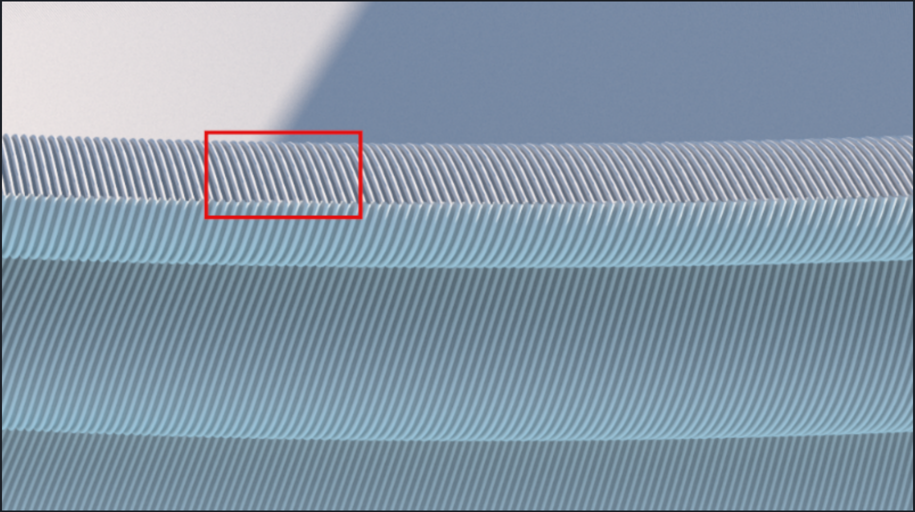

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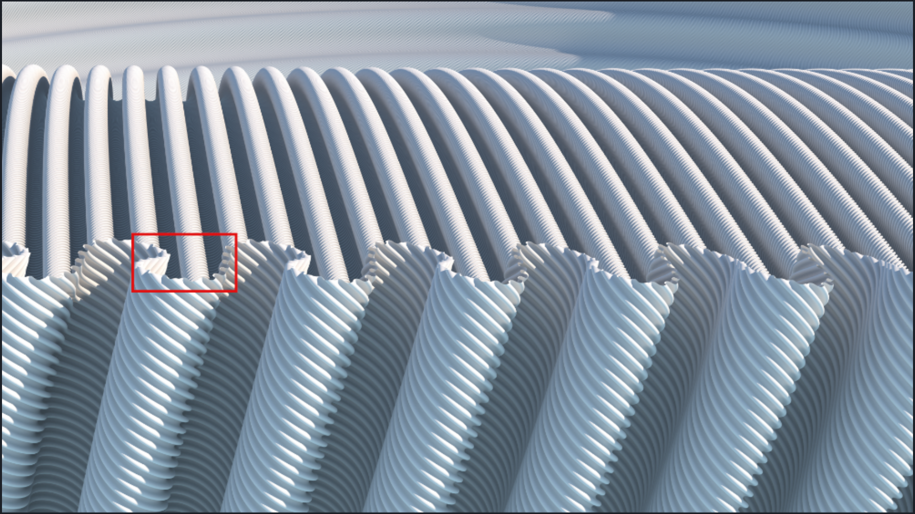

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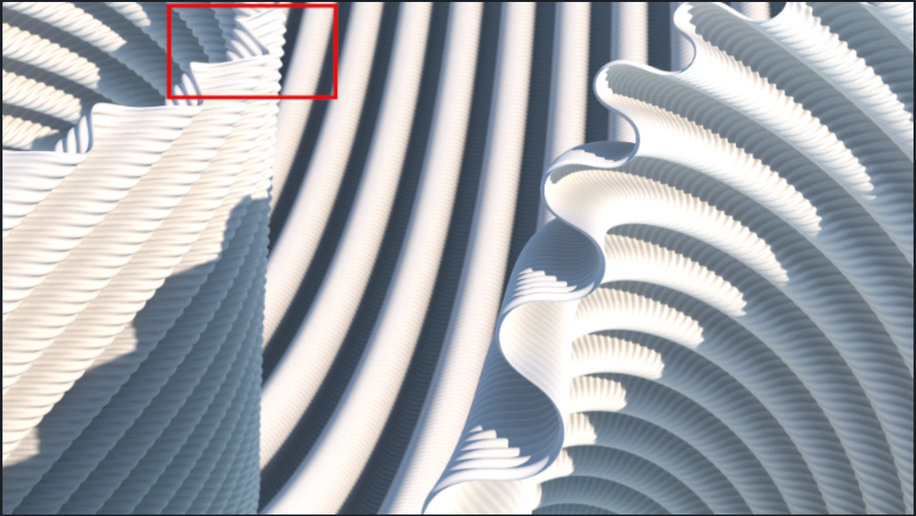

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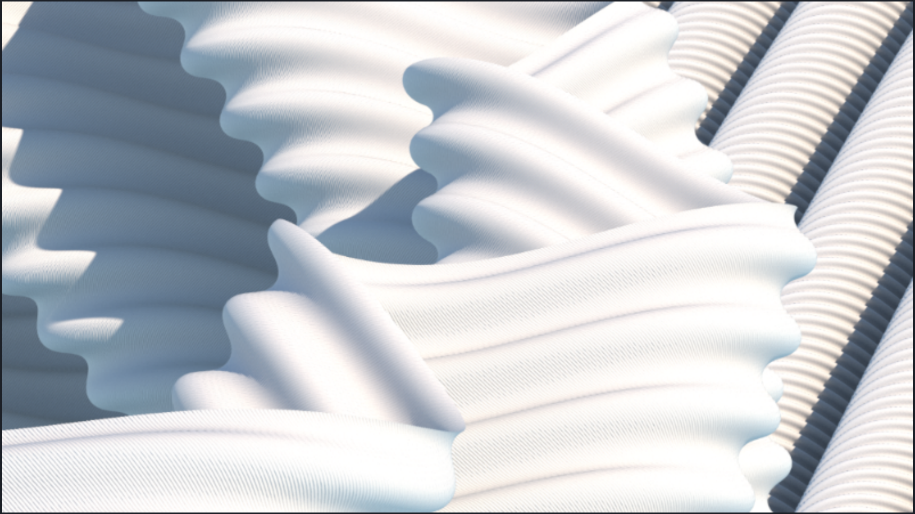

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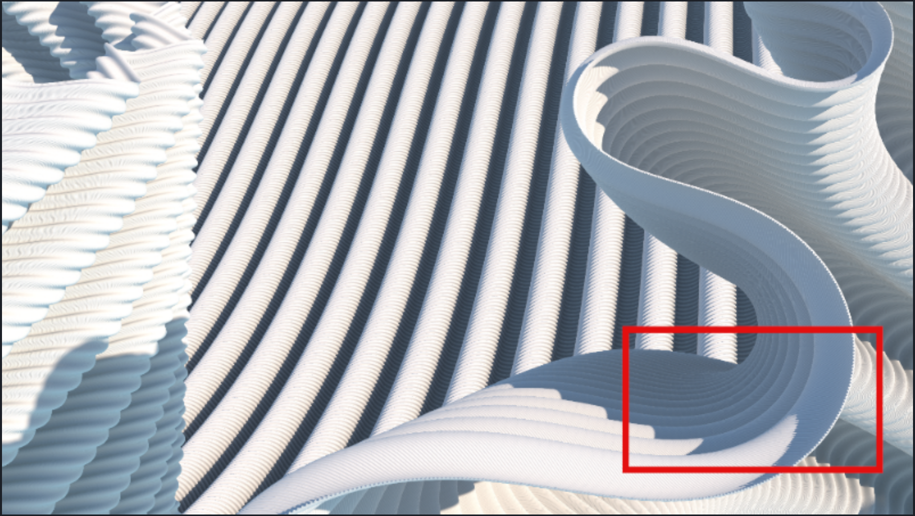

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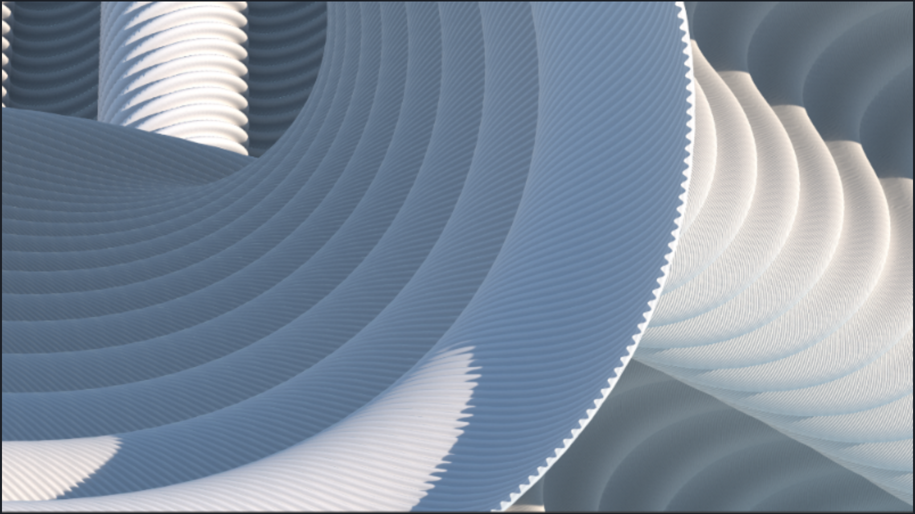

The analytic expression of the "edge curve" is similar to a lacunar Fourier series.

Definition

Let M be a non-compact manifold and $f : M \to \mathbb{R}^n$ be a map.
Let $(x_k)_k$ be a divergent sequence of points of M. If the sequence $(f(x_k))_k$ converges in $\mathbb{R}^n$, its limit is called a limit point of f. The set of limit points is denoted by $L(f)$.

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Question: By modifying parameters of the construction (initial map, way to reduce the isometric default, ...), what can we have as limit set ?

The limit set

Theorem (De Lellis, 2017)

Let $f_0 : (M, g) \to \mathbb{E}^n$ be a strictly short embedding, then there exists a C^1-isometric embedding $f : (M, g) \to \mathbb{E}^n$ with the same limit set

$$L(f) = L(f_0)$$

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$$L(f) = L(f_0)$$

An amazing corollary

Let's imagine a short embedding f_0 of the hyperbolic plane such that

$L(f_0)$ is reduced to a point.

Then there exists a C^1-isometric embedding f of $\mathbb{H}^2$ such that $f(\mathbb{H}^2) \cup L(f)$ is homeomorphic to a sphere, so a hyperbolic sphere.

The limit set

Theorem (Borrelli-Lazarus-T.-Thibert)

Let Γ be any smooth immersed closed curve in $\mathbb{E}^3$, then there exists a C^1 isometric immersion of $f : \mathbb{H}^2 \to \mathbb{E}^3$ whose limit set $L(f)$ is Γ.

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Remark 1. Circles in $\mathbb{H}^2$ can be chosen with an arbitrarily large perimeter, however the length of $L(f)$ is finite: the length map is lower semi-continuous.

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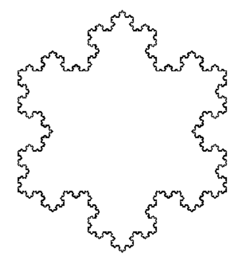
Remark 2. Any point of $L(f)$ is at infinite distance of any points of the surface for the induced distance of $\mathbb{E}^3$.

Theorem (Borrelli-Lazarus-T.-Thibert)

Let Γ be any planar Jordan curve, then there exists a C^1 isometric immersion of $f : \mathbb{H}^2 \to \mathbb{E}^3$ whose limit set $L(f)$ is Γ.



The Koch snowflake is a Jordan curve.

Thanks for your attention!