Computing the length spectrum of combinatorial graphs on the torus

Matthijs Ebbens

Joint work with Francis Lazarus

Université Grenoble Alpes

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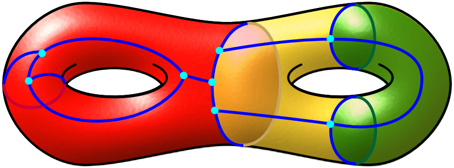
Contents

Combinatorial surfaces



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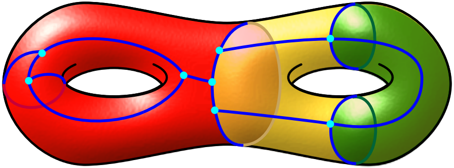

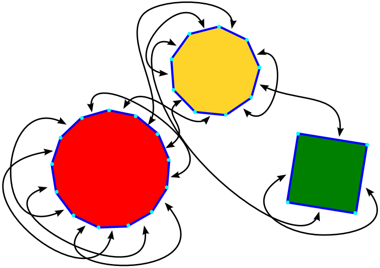

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- The length of a path is sum of the weights of its edges.
- A cycle is a closed walk.

Length spectrum

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Length spectrum

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- The length spectrum is the ordered sequence of lengths of free homotopy classes.

Motivation

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- There exist non-isometric hyperbolic surfaces which have the same length spectrum [Vignéras (1980)],
- but only finitely many [McKean (1972)]
- and they are very rare [Wolpert (1979)].

Problem statement

Goal: Find an algorithm which, given a weighted graph G cellularly embedded on a surface of genus g and a positive integer k, computes the first k values of the length spectrum of G.

Results for the systole ($k = 1$)

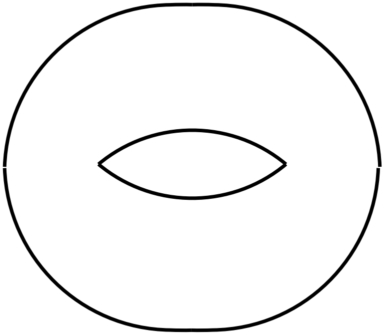
- [Thomassen (1990)]: unweighted, $O(n^3)$
- [Erickson, Har-Peled (2004)]: weighted, $O(n^2 \log n)$.
- [Cabello, Chambers (2007)]: weighted, $O(g^3 n \log n)$.
- [Cabello, Colin de Verdière, Lazarus (2012)]: unweighted, $O(gn|\ell_1|)$.

Computing the length spectrum of graphs on the torus

Theorem

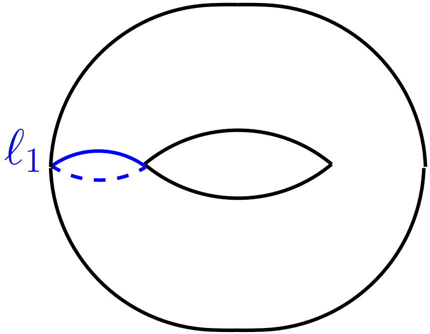
The first k values of the length spectrum of a weighted graph G of complexity n cellularly embedded on the torus can be computed in time $O(kn^2 \log(kn))$.

Algorithm



Algorithm

Compute the shortest non-contractible cycle ℓ_1 [Erickson, Har-Peled (2004)].



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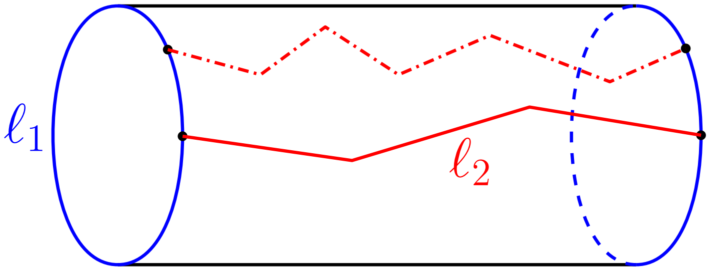

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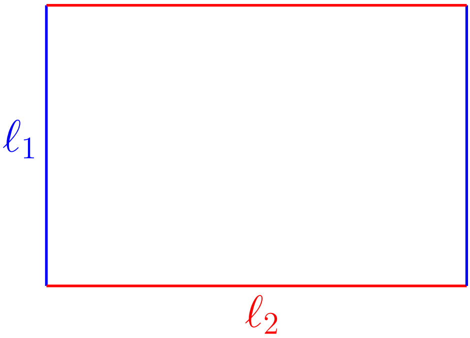

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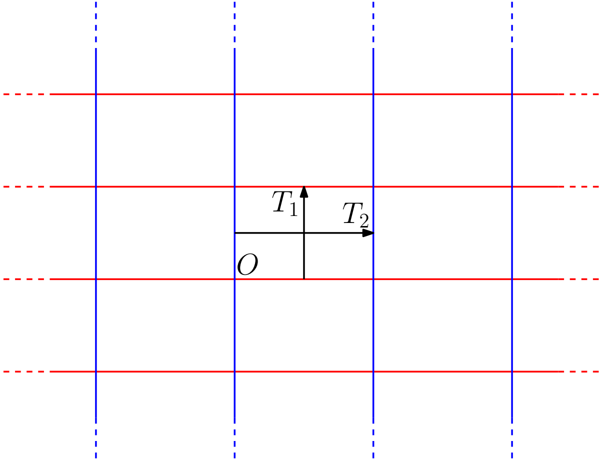
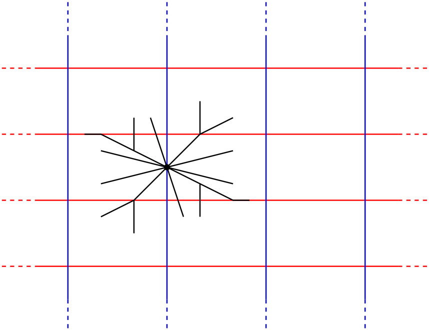


Algorithm

- Compute the length spectrum of loops based at a vertex v using Dijkstra's shortest path algorithm with source v.



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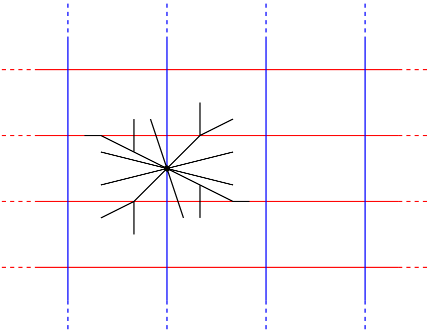

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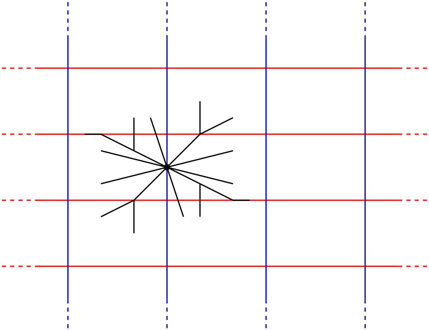
- Repeat for the other vertices and sort.
- Note: sufficient to consider only the vertices on ℓ_1.



Complexity I

Lemma

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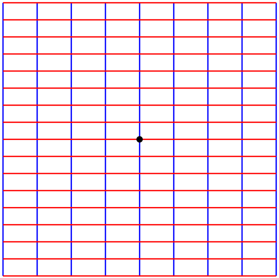
Equivalent: there are $\Omega(r^2|\ell_1|^{-1}|\ell_2|^{-1})$ translates within distance r of v.

Complexity I: sketch of proof

To show: there are $\Omega(r^2|\ell_1|^{-1}|\ell_2|^{-1})$ translates within distance r of v.



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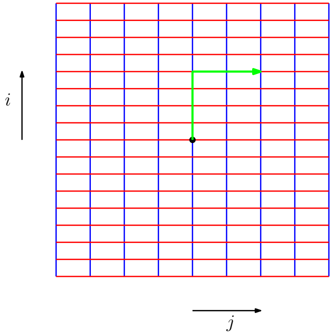

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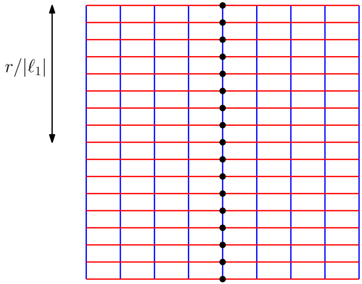
- $d(O, T_1^i T_2^j(O)) \leq |i||\ell_1| + |j||\ell_2|$.
- If $j = 0$ and $|i| \leq r/|\ell_1|$, then $d(O, T_1^i T_2^j(O) \leq r$.



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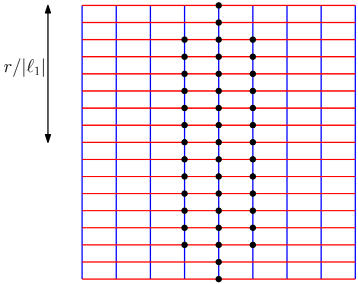

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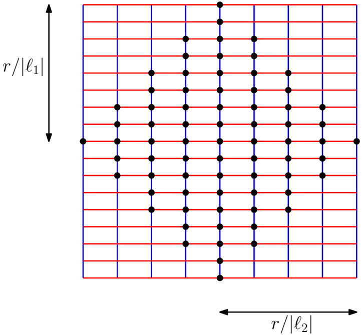
- Continue until $j = \lfloor r/|\ell_2| \rfloor$.



Complexity II

Lemma

The number of vertices within distance $r \in \mathbb{R}_{>0}$ from a given vertex v is $O(nr^2|\ell_1|^{-1}|\ell_2|^{-1})$.

Note: n is the complexity of the graph.

Complexity II: proof

Lemma

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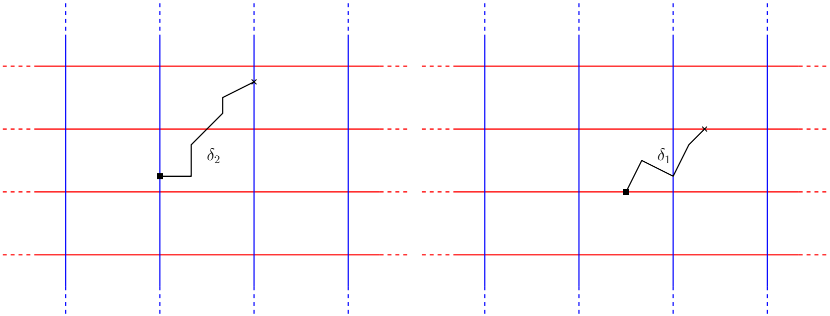
- Claim 1: the distance between vertical lines is at least $\frac{1}{2}|\ell_2|$.
- Claim 2: the distance between horizontal lines is at least $\frac{1}{2}|\ell_1|$.



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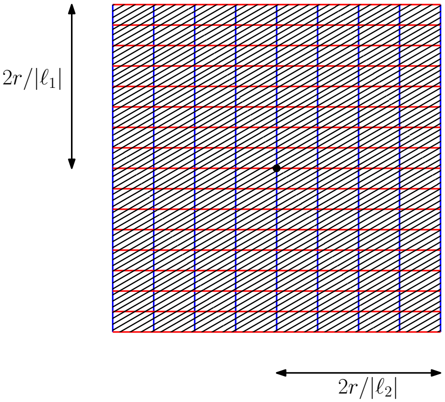

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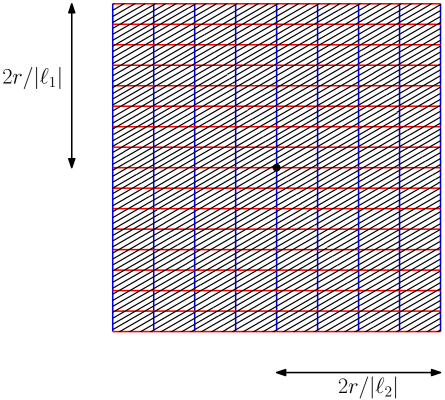

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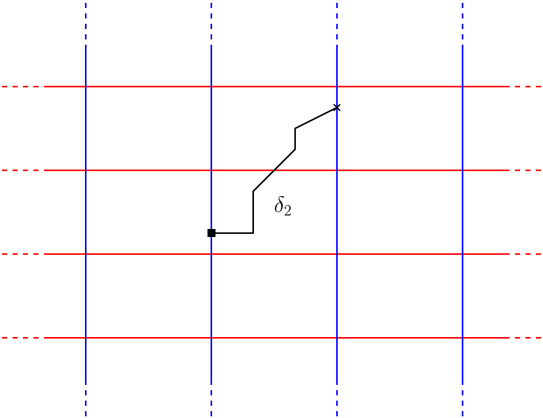

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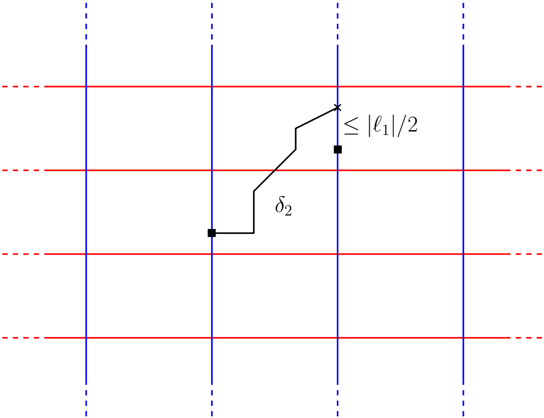

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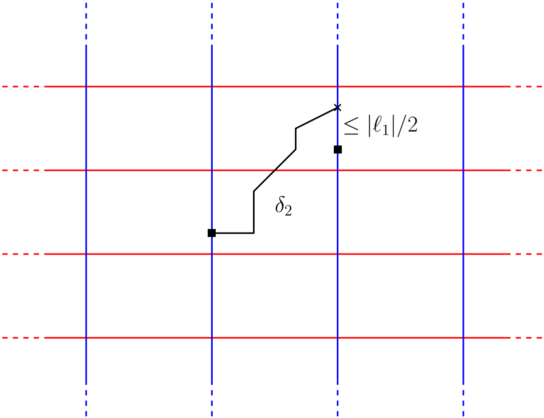

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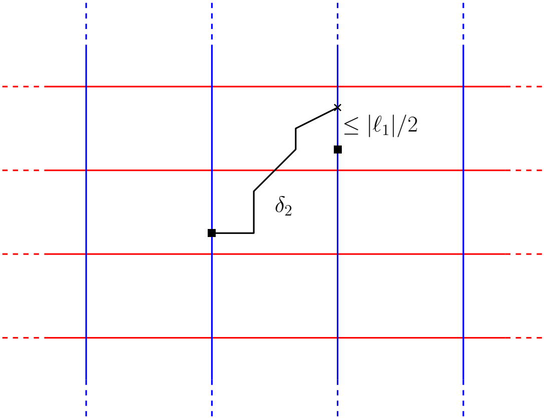

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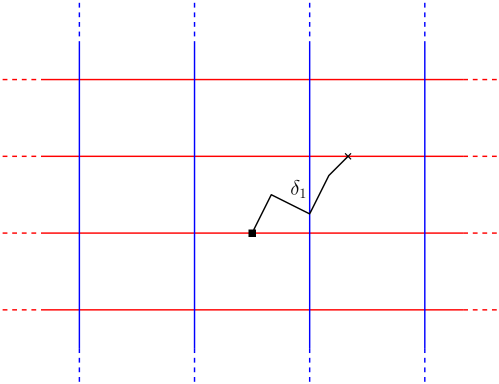

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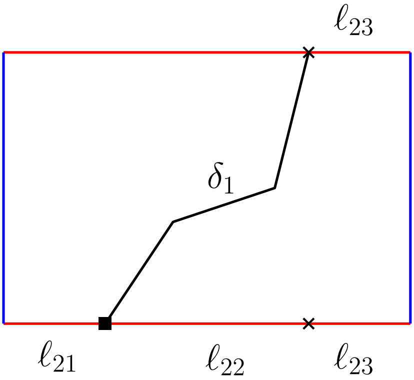

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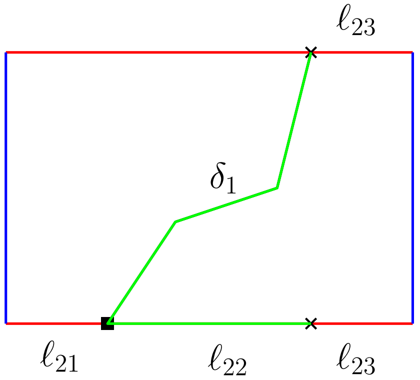

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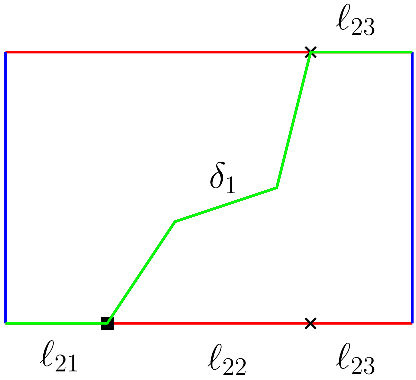

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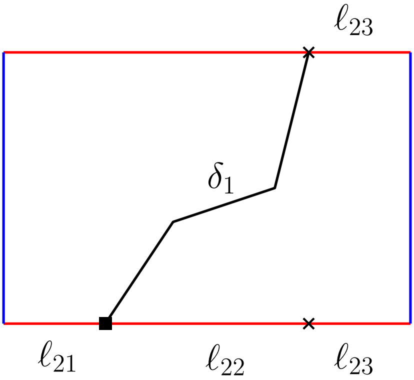

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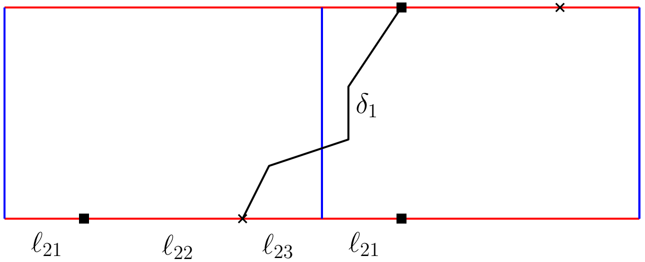

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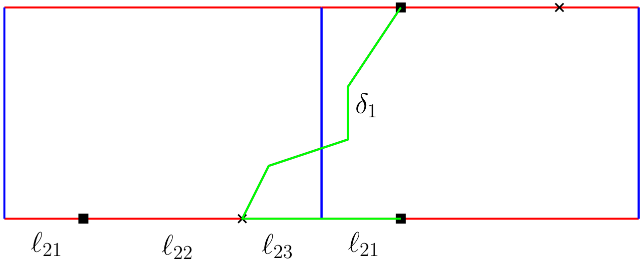

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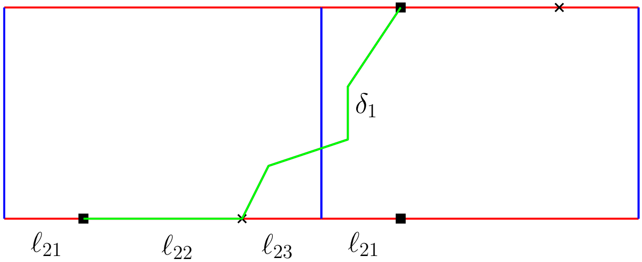

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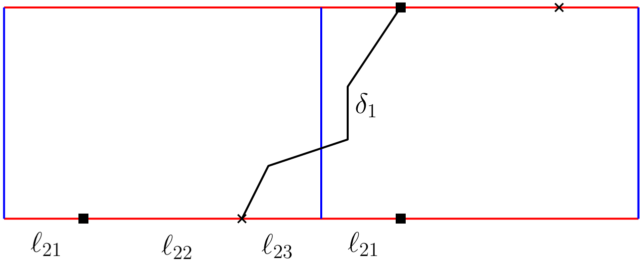

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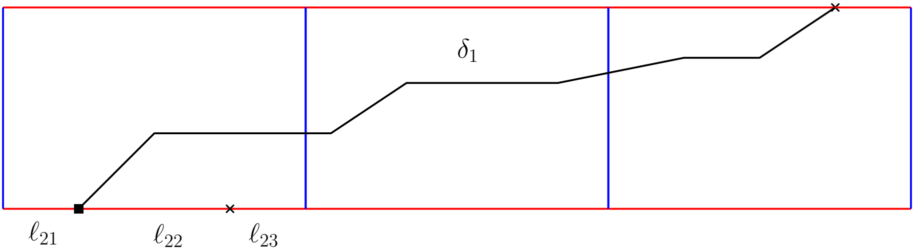

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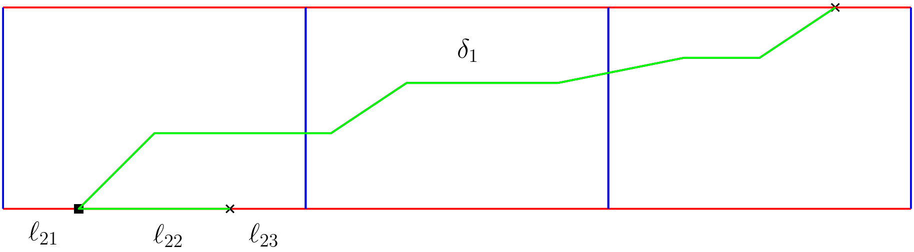

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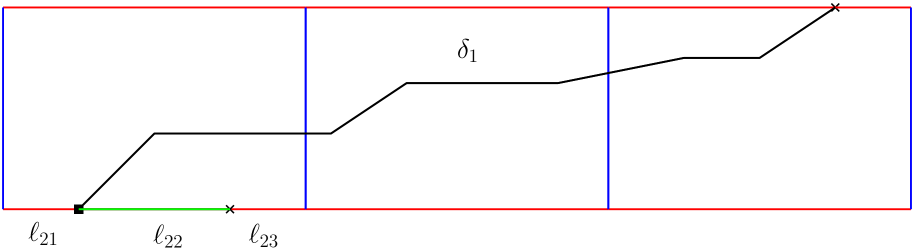

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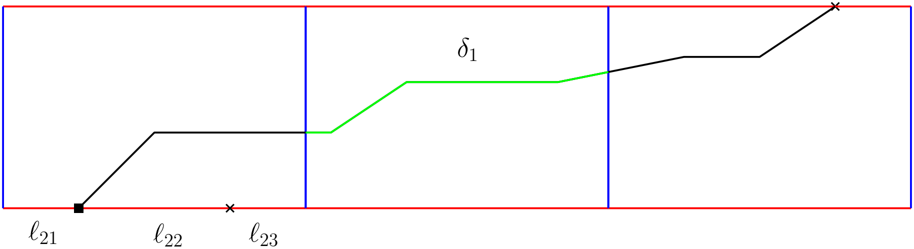

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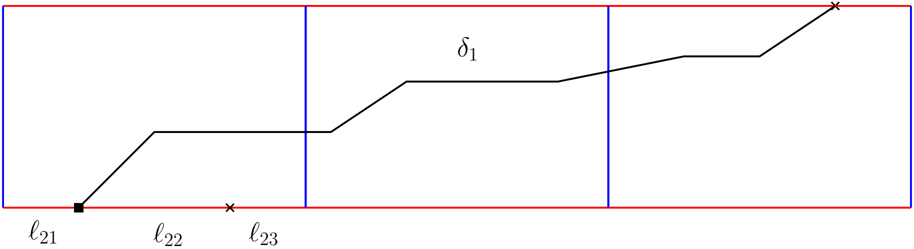

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The $2k$ closest translates of a vertex v on ℓ_1 have distance $O(\sqrt{k|\ell_1||\ell_2|})$ from v.

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- It is not clear how $\ell_1, \ell_2, \ldots, \ell_{2g}$ should be chosen now.
- $\pi_1(S)$ is no longer commutative, so keeping track of the distance between the source and translates will be more complicated.

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- It is not clear if we can show that the distance between translates of a side is "at least $\frac{1}{2}|\ell_i|$".
- Even if we can show something like that, we end up with a factor 2^{2g}.

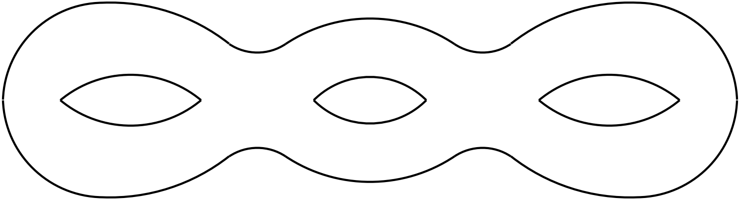
Computing a second systole

Theorem

A second systole of a weighted graph G of complexity n cellularly embedded on a surface S of genus g can be computed in time $O(n^2 \log n)$.

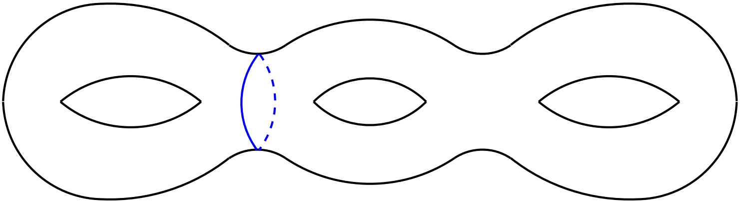
Algorithm for computing a second systole

Compute a shortest non-contractible cycle ℓ_1 [Erickson, Har-Peled (2004)].



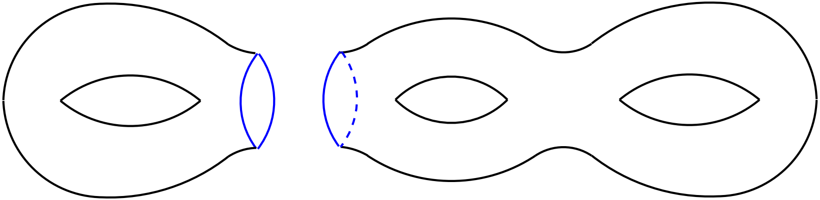
Algorithm for computing a second systole

If ℓ_1 is separating:



Algorithm for computing a second systole

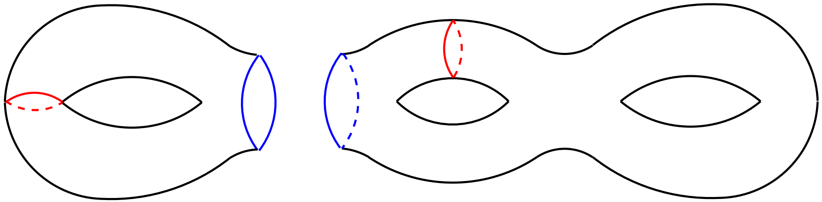
If ℓ_1 is separating:



Algorithm for computing a second systole

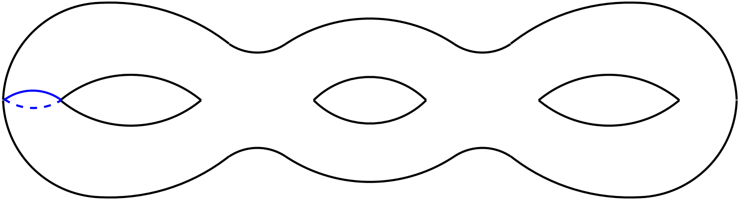
Second systole is the shortest of:

- the shortest essential cycle in both components [Erickson, Worah (2010)],
- ℓ_1^2: the cycle obtained by traversing ℓ_1 twice.



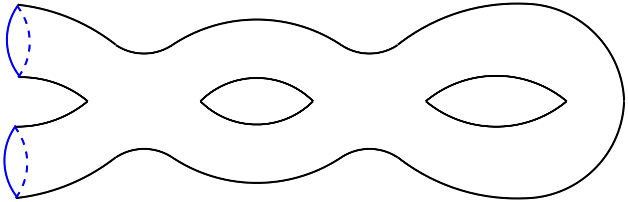
Algorithm for computing a second systole

If ℓ_1 is non-separating:



Algorithm for computing a second systole

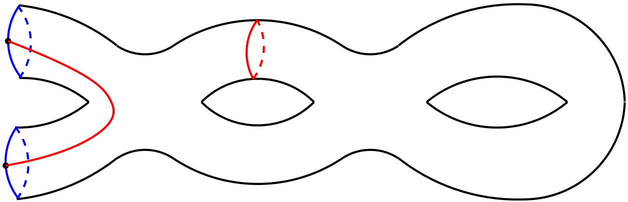
If ℓ_1 is non-separating:



Algorithm for computing a second systole

Second systole is the shortest of:

- the shortest essential cycle [Erickson, Worah (2010)],
- the shortest path between corresponding vertices on the boundary components,
- ℓ_1^2: the cycle obtained by traversing ℓ_1 twice.



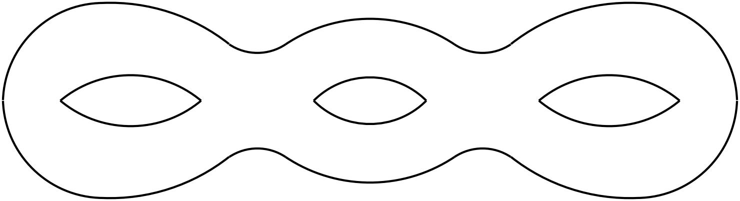
Computing a third systole

Theorem

A third systole of a weighted graph G of complexity n cellularly embedded on a surface S of genus g can be computed in time $O(n^2 \log n)$.

Algorithm for computing a third systole

Compute a shortest non-contractible cycle ℓ_1 [Erickson, Har-Peled (2004)] and second systole ℓ_2.



Algorithm for computing a third systole

- Case 1: $\ell_2 = \ell_1^2$,
- Case 2: $\ell_2 \neq \ell_1^2$, ℓ_1 and ℓ_2 are separating,
- Case 3: $\ell_2 \neq \ell_1^2$, ℓ_1 is separating and ℓ_2 is non-separating (or reversely),
- Case 4: $\ell_2 \neq \ell_1^2$, ℓ_1 and ℓ_2 are non-separating and do not cross,
- Case 5: $\ell_2 \neq \ell_1^2$, ℓ_1 and ℓ_2 are non-separating and cross.

Case 1

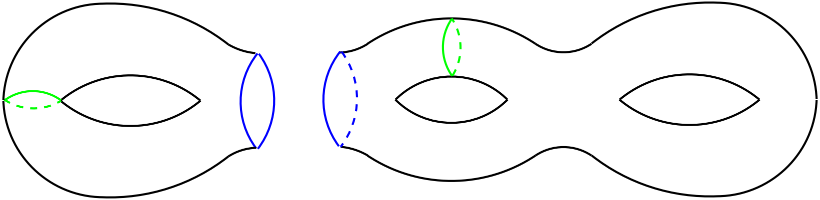
If $\ell_2 = \ell_1^2$, use the algorithm for the second systole:

Case 1

If $\ell_2 = \ell_1^2$, use the algorithm for the second systole:

If ℓ_1 is separating, then a third systole is the shortest of:

- the shortest essential cycle in both components [Erickson, Worah (2010)],
- ℓ_1^3: the cycle obtained by traversing ℓ_1 three times.

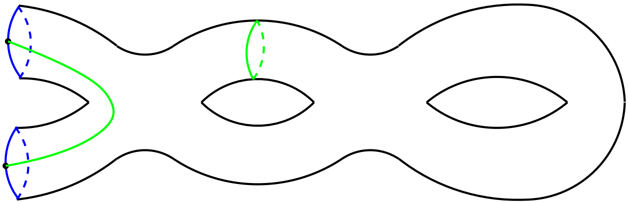


Case 1

If $\ell_2 = \ell_1^2$, use the algorithm for the second systole:

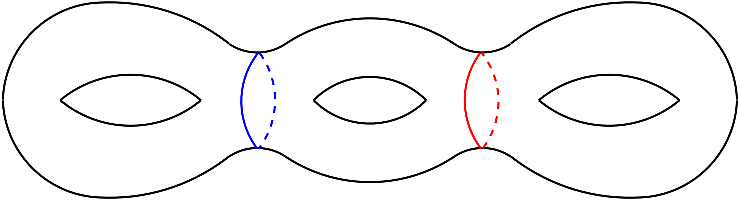
If ℓ_1 is non-separating, then a third systole is the shortest of:

- the shortest essential cycle [Erickson, Worah (2010)],
- the shortest path between corresponding vertices on the boundary components,
- ℓ_1^3: the cycle obtained by traversing ℓ_1 three times.



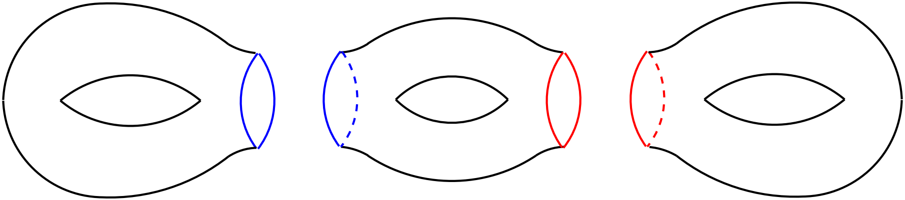
Case 2

If ℓ_1 and ℓ_2 are separating:



Case 2

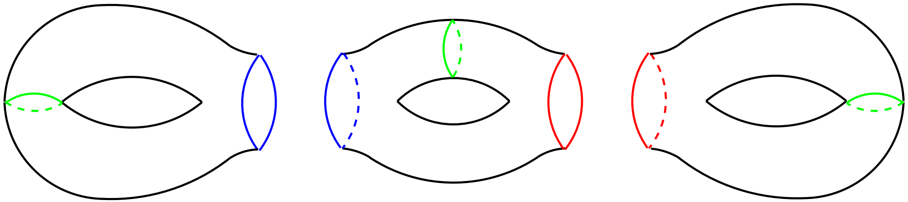
If ℓ_1 and ℓ_2 are separating:



Case 2

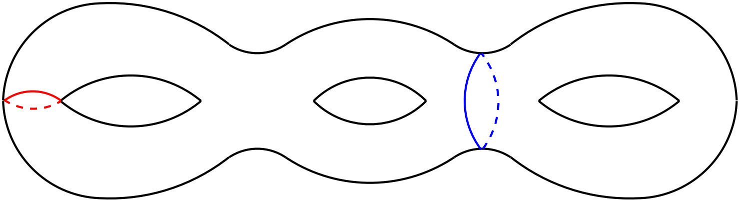
Third systole is the shortest of:

- the shortest essential cycle in all three components [Erickson, Worah (2010)],
- ℓ_1^2: the cycle obtained by traversing ℓ_1 twice.



Case 3

If ℓ_1 is separating and ℓ_2 is non-separating (or the other way around):



Case 3

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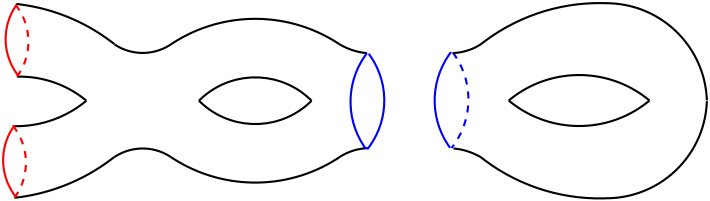
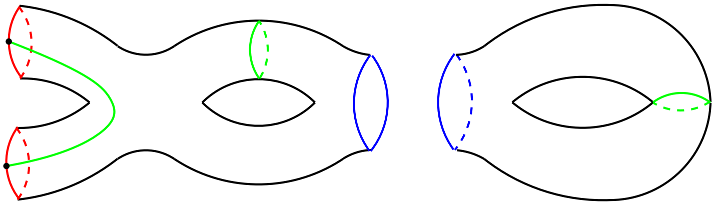


Case 3

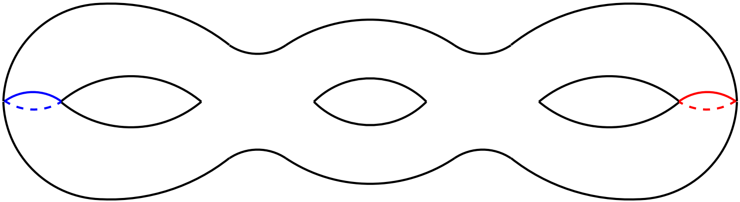
Third systole is the shortest of:

- the shortest essential cycle in both components [Erickson, Worah (2010)],
- the shortest path between corresponding vertices on the boundary components,
- ℓ_1^2: the cycle obtained by traversing ℓ_1 twice.



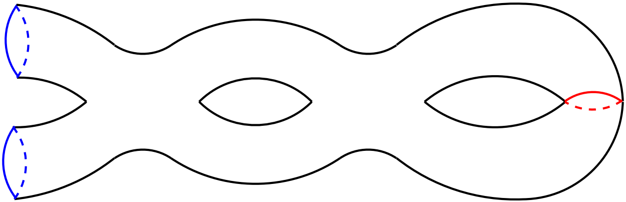
Case 4

If ℓ_1 and ℓ_2 are both non-separating and do not cross:



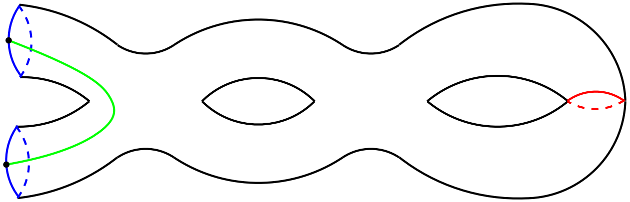
Case 4

If ℓ_1 and ℓ_2 are both non-separating and do not cross:



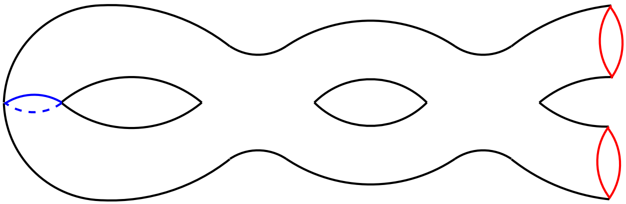
Case 4

If ℓ_1 and ℓ_2 are both non-separating and do not cross:



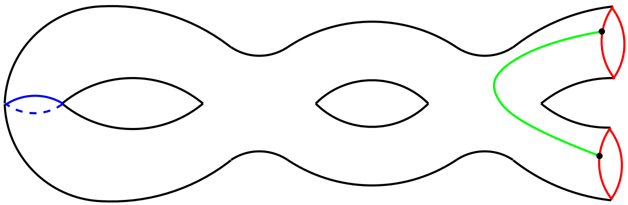
Case 4

If ℓ_1 and ℓ_2 are both non-separating and do not cross:



Case 4

If ℓ_1 and ℓ_2 are both non-separating and do not cross:



Case 4

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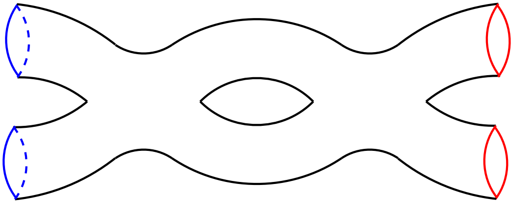
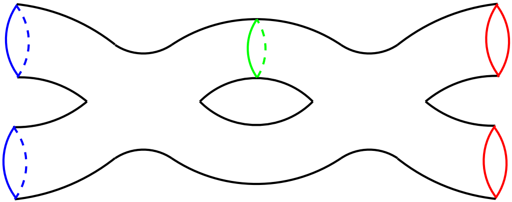


Case 4

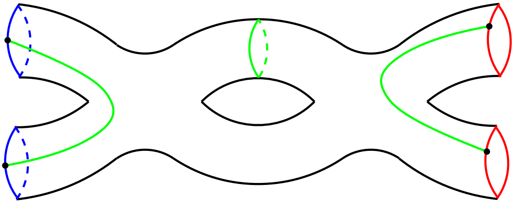
If ℓ_1 and ℓ_2 are both non-separating and do not cross:



Case 4

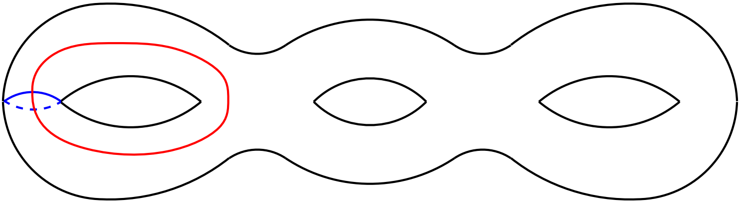
Third systole is the shortest of:

- the shortest essential cycle [Erickson, Worah (2010)],
- the shortest path between corresponding vertices on the boundary components,
- ℓ_1^2: the cycle obtained by traversing ℓ_1 twice.



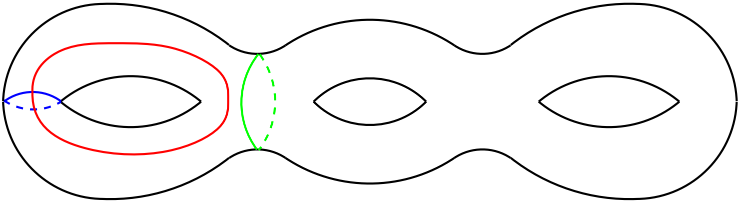
Case 5

If ℓ_1 and ℓ_2 are both non-separating and cross:



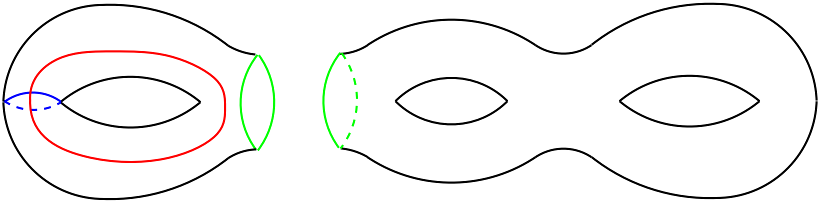
Case 5

Compute a shortest cycle homotopic to $\ell_1 \cdot \ell_2$ [Colin de Verdière, Erickson (2010)].



Case 5

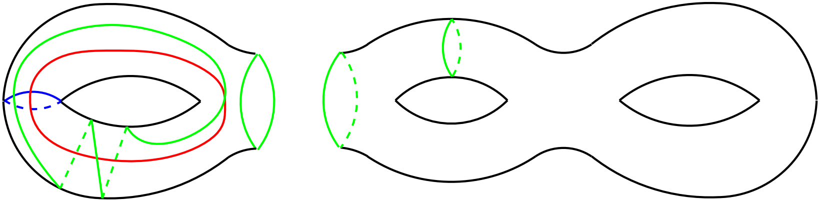
Compute a shortest cycle homotopic to $\ell_1 \cdot \ell_2$ [Colin de Verdière, Erickson (2010)].



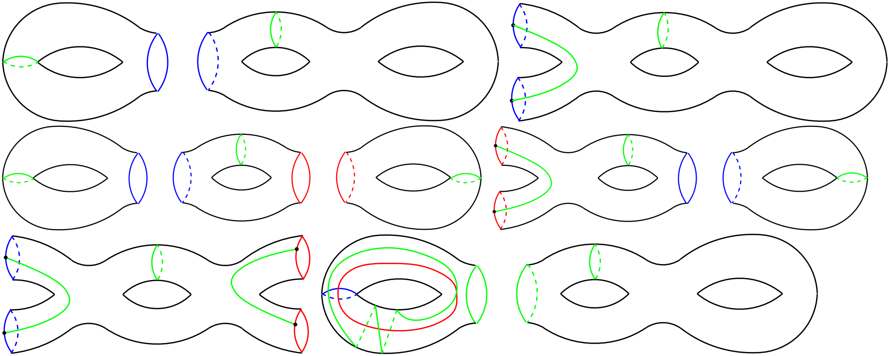
Case 5

Third systole is the shortest of:

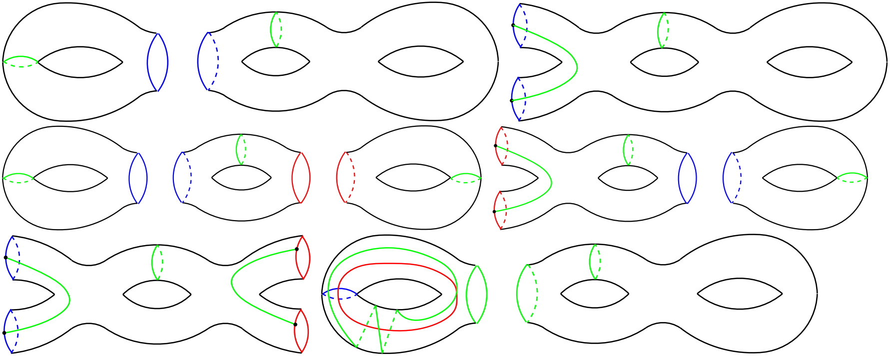
- the shortest essential cycle in the right component [Erickson, Worah (2010)],
- the third shortest cycle in the left component,
- the boundary curve,



Third systole



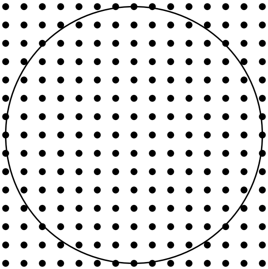
Third systole



Next values of the length spectrum even more cases?

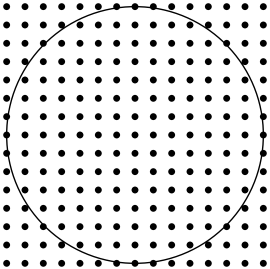
Gauss circle problem

- Question: how many integer lattice points are there in a circle of radius r centered at the origin?



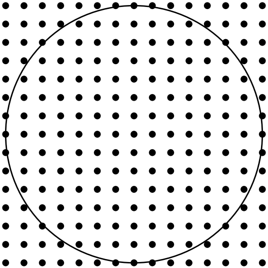
Gauss circle problem

- Question: how many integer lattice points are there in a circle of radius r centered at the origin?
- Answer: $\sim \pi r^2$.



Gauss circle problem

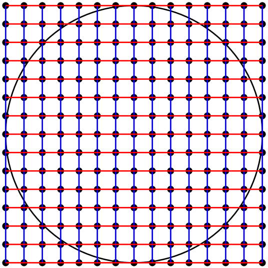
- Question: how many integer lattice points are there in a circle of radius r centered at the origin?
- Answer: $\sim \pi r^2$.
- More general answer: $\sim \dfrac{\pi r^2}{\text{area(F)}}$.



Relation with torus

Lemma

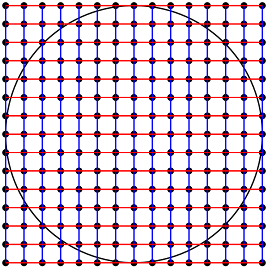
The number of translates of a vertex v on ℓ_1 within distance r of v is $\Omega(r^2|\ell_1|^{-1}|\ell_2|^{-1})$.



Relation with torus

Lemma

The number of vertices within distance $r \in \mathbb{R}_{>0}$ from a given vertex v is $O(nr^2|\ell_1|^{-1}|\ell_2|^{-1})$.



Hyperbolic lattice point problem

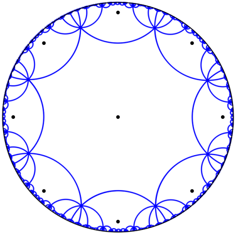
Theorem (Huber (1956))

Γ *Fuchsian group such that* $\mathbb{H}/\Gamma$ *is a closed hyperbolic surface of genus* g.

$$N(r, z, z_0) := \#\{T \in \Gamma \mid d_{\mathbb{H}}(z_0, T(z)) \leq r\}$$

Then

$$N(r, z, z_0) \sim \frac{e^r}{4\pi(g-1)}.$$



Lattice point problem in graphs?

$\tilde{G}$ infinite periodic weighted graph embedded on the universal cover of S, where $\pi_1(S)$ is the group of covering transformations.

$$N(r, v, v_0) := \#\{T \in \pi_1(S) \mid d_{\tilde{G}}(v_0, T(v)) \leq r\}$$

Question: is it true that

$$N(r, v, v_0) \sim \frac{\text{area}(B_r(v_0))}{\text{area}(F)}?$$

Or weaker, is it true that

$$N(r, v, v_0) \sim N(r, v, v_1)?$$