Experimental analysis of Delaunay flip algorithms on genus two hyperbolic surfaces

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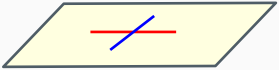
Gamble team, Loria, INRIA (Nancy)

Next slides

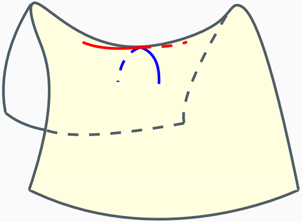
1. Background

- Hyperbolic plane
- Closed oriented genus 2 hyperbolic surfaces
- Delaunay triangulations in this context

Poincaré disk model of hyperbolic plane

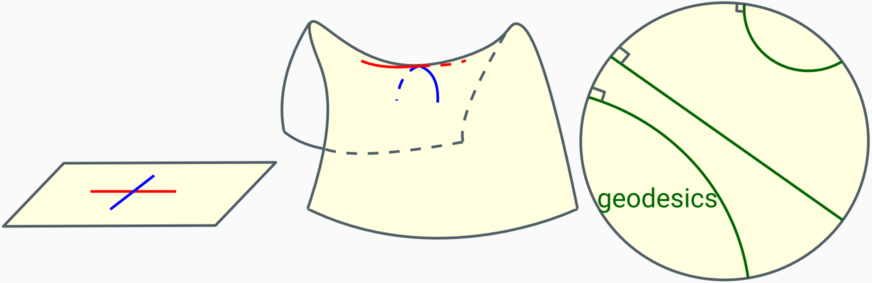


Euclidean



Hyperbolic

Poincaré disk model of hyperbolic plane



Euclidean

Hyperbolic

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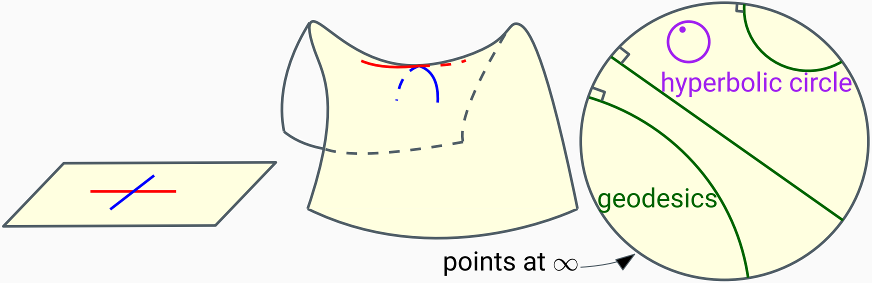

Euclidean

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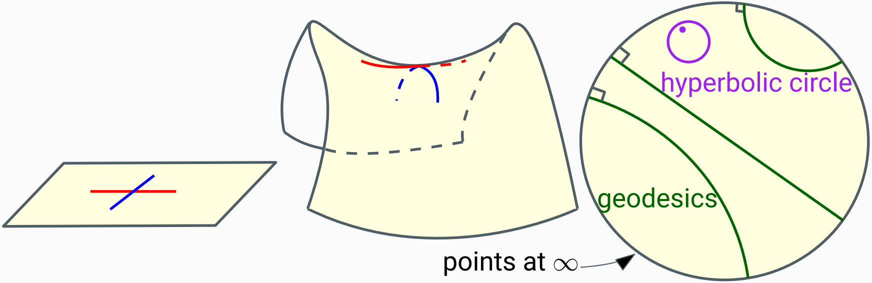

Euclidean

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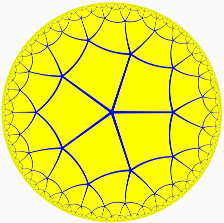
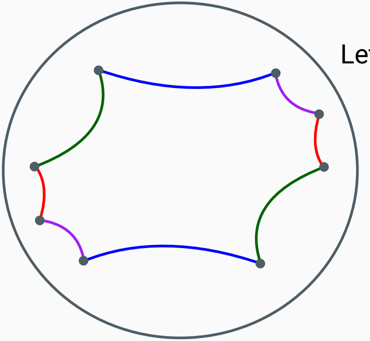
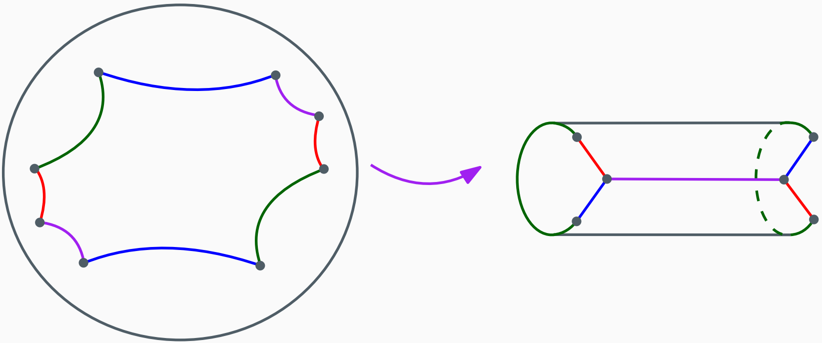


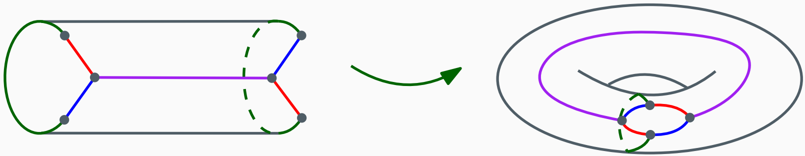


Let’s build a surface from an octagon!

Closed oriented genus 2 hyperbolic surfaces from octagons



Closed oriented genus 2 hyperbolic surfaces from octagons



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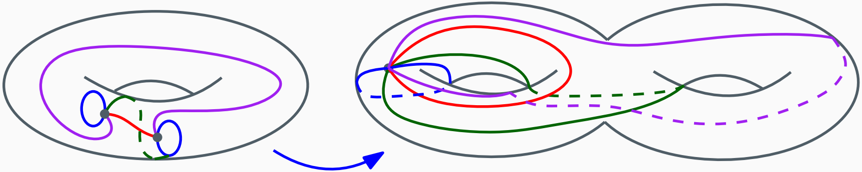
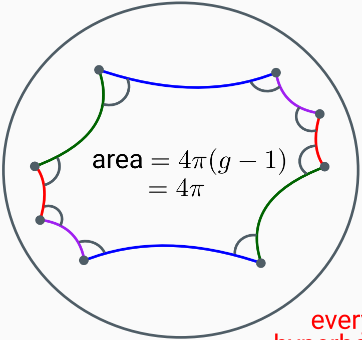
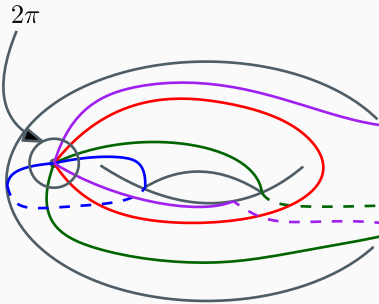


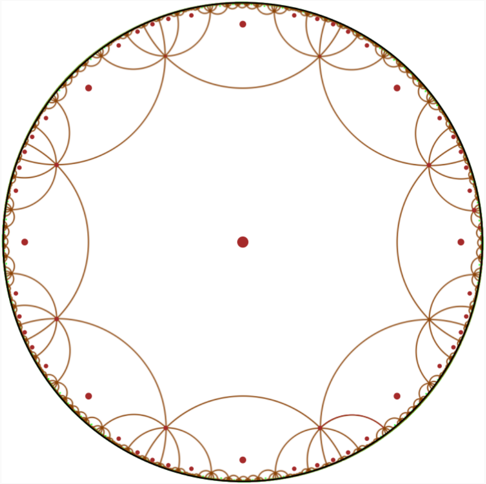
Closed oriented genus 2 hyperbolic surfaces from octagons

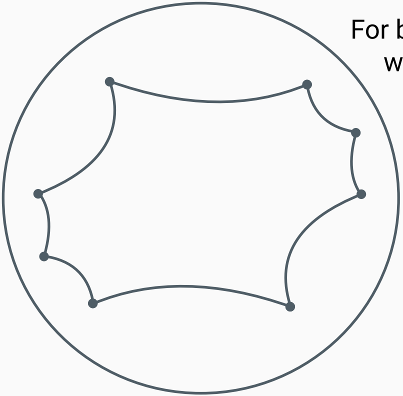




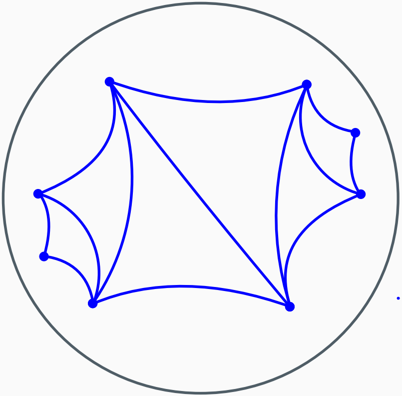
every closed oriented $g = 2$
hyperbolic surface obtained this way

Lifts



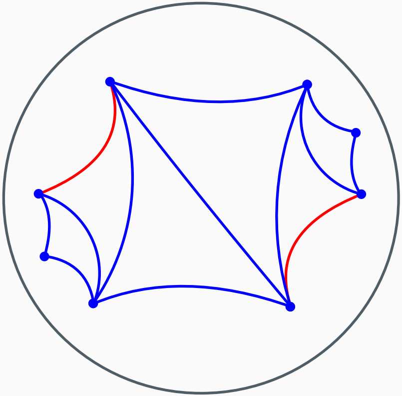


For building a triangulation
we start from a "valid" octagon . . .

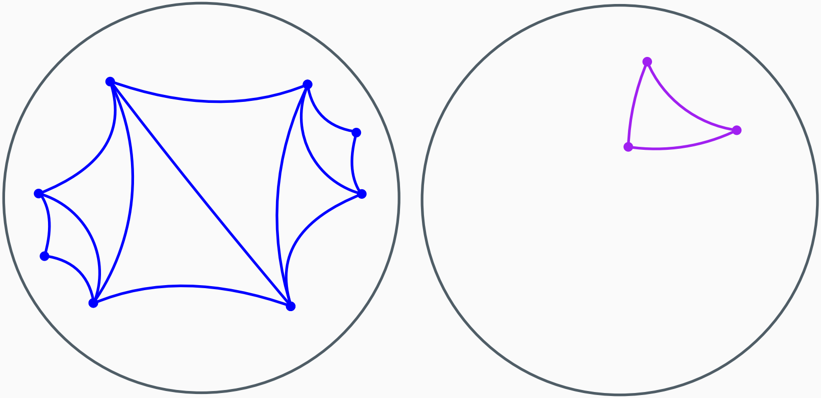


…and triangulate the octagon

Delaunay triangulations in this context

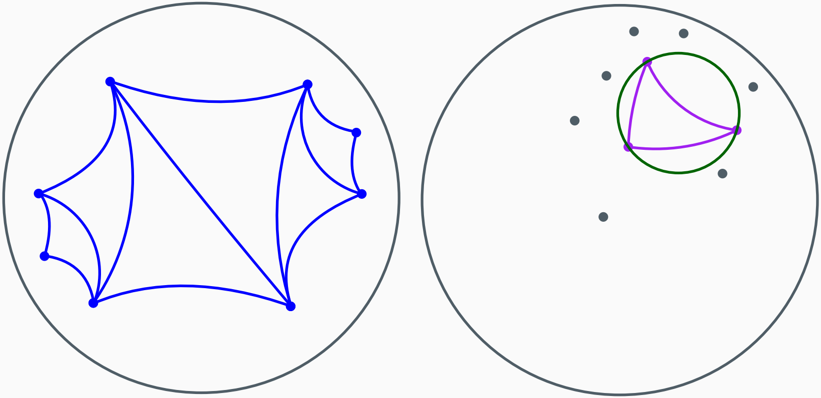


remember, those two
segments
are in fact the same edge
of the triangulation



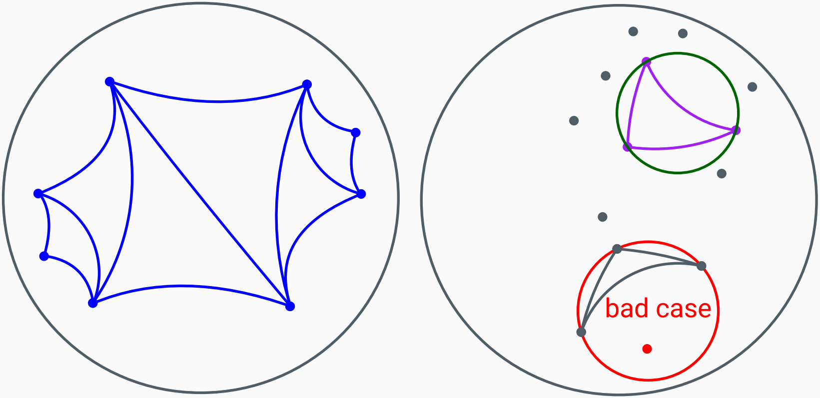
Delaunay is as usual : for every lift of a face . . .

Delaunay triangulations in this context



Delaunay is as usual : for every lift of a face . . .

. . . ask for every lift of a point to be outside the circumdisk



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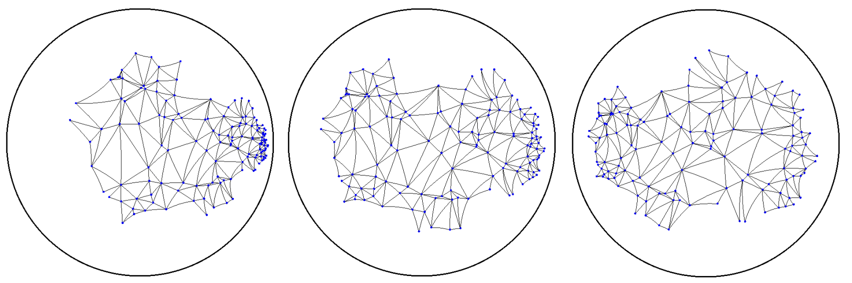
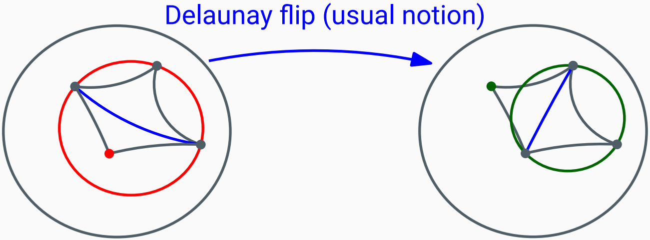


3 representations of a single Delaunay triangulation

Next slides

2. Our experiments

- Delaunay flip algorithm
- Motivations
- Practical constraints
- Experiments and results


Delaunay flip (usual notion)

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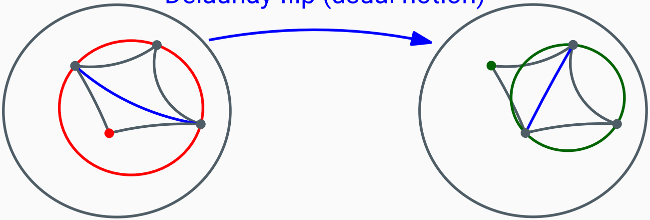


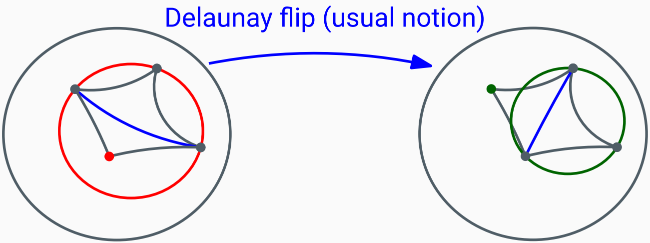
finishes after $O(\Delta^{6g-4} \cdot n^2)$ flips

$\rightarrow$ Despré, Schlenker, Teillaud (SoCG'20)

$\Delta \simeq$ diameter of the octagon
"stretching factor"
interesting parameter

genus $g = 2$

nb vertices $n = 1$
dependency in n is tight
algorithm useful for small n only



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genus $g = 2$

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dependency in n is tight
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Our contribution ($g = 2, n = 1$)

- Experiments using rational numbers only (thanks to a density result)

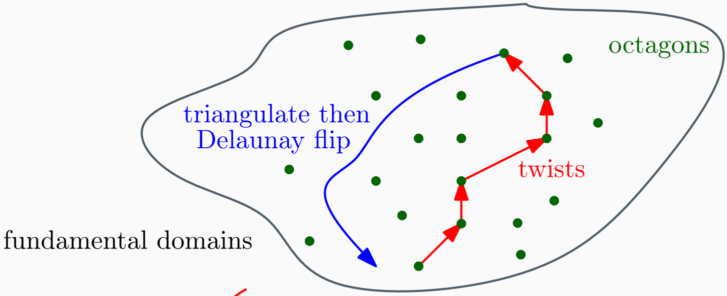
Motivation

OBJECTIVE :

Use Delaunay flips to go from "bad" to "good" fundamental domain (hence $n = 1$)

WHAT WE DID :

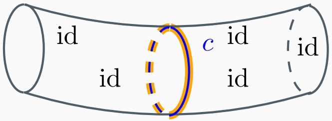
We compared red path and blue path

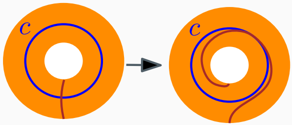


↪ this is a walk in the mapping class group ...

Motivation

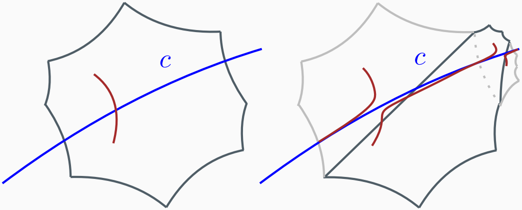
Dehn twists are particular isotopy classes of homeomorphisms





they generate the mapping class group

isotopy classes of homeos + "composition"



Octagon	Twist of octagon
↓	↓
Isotopy class of homeos	Composition by a Dehn twist

mapping class group $\longleftrightarrow$ Delaunay flips ?

[!] We explore only a subgroup of the mapping class group

Practical constraints

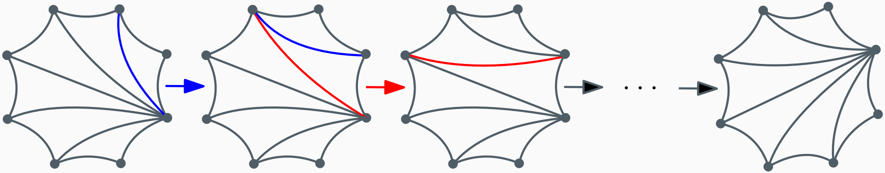
(we only need exact predicates for robustness,
but predicates involve constructions)

We must match:

- exact number type
- efficient computations

(CORE::Expr)

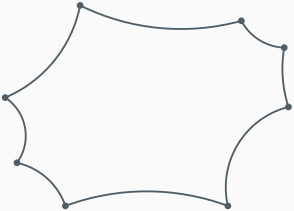
Experiment : flips implemented with exact algebraic numbers



Result : Computations are inefficient (too slow)

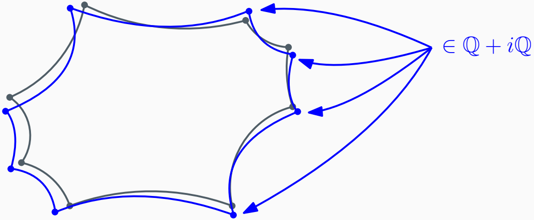
! Thus rational numbers are our friends !

start with an "almost valid" octagon

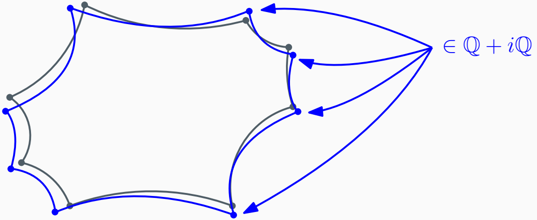


Experiments

start with an "almost valid" octagon



(step 1/3) we construct a "rational" and "valid" octagon nearby start with an "almost valid" octagon



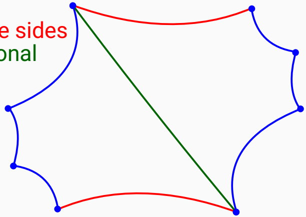
(step 1/3) we construct a "rational" and "valid" octagon nearby

gives octagons with small diameter only

$\longrightarrow$ (step 2/3) we apply twists ...

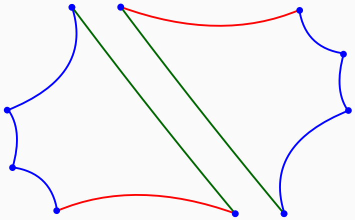
Experiments

choose
a pair of opposite sides
and a diagonal



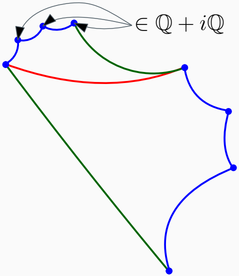
Experiments

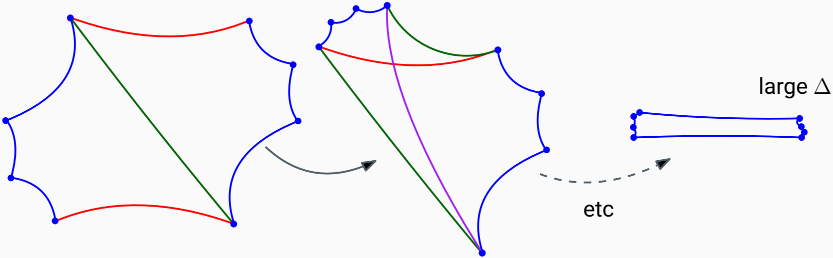
we cut along the diagonal



Experiments

and glue the paired sides



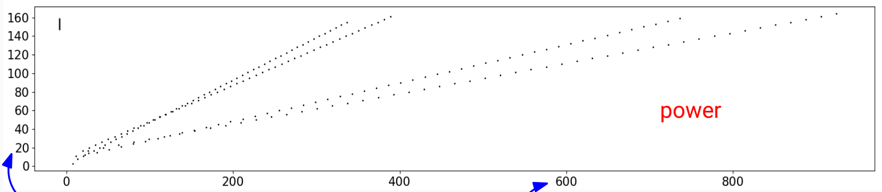


Choosing paired sides

power : always the "same" pair

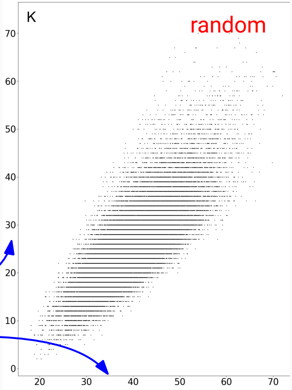
random : uniformly random

Results



flips

diameter



Triangulations . . .
$\rightarrow$ of the same surface
$\rightarrow$ with same unique vertex

(step 3/3) Apply the Delaunay flip algorithm

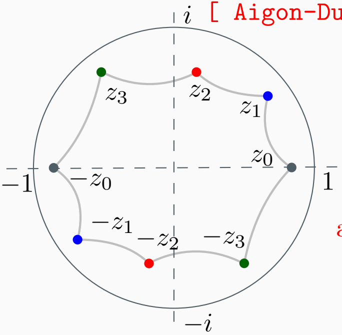
flips

10 $\ln$ (diameter)

3. Details

- Generation of rational octagons
- Representation of triangulations (data structure)

[Aigon-Dupuy, Buser, et al. Math. Phys.'05]



Input: z_1, z_2, z_3 such that

$$0 < \arg z_1 < \arg z_2 < \arg z_3 < \pi$$

Output: (if exists) $z_0 \in]0, 1[$ such that

$$\text{hyperbolic area of octagon} = 4\pi$$

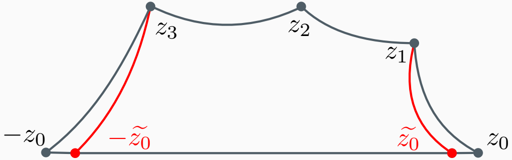
all closed oriented genus 2 hyperbolic surfaces obtained this way

Problem : $z_1, z_2, z_3 \in \mathbb{Q} + i\mathbb{Q} \not\Rightarrow z_0 \in \mathbb{Q}$

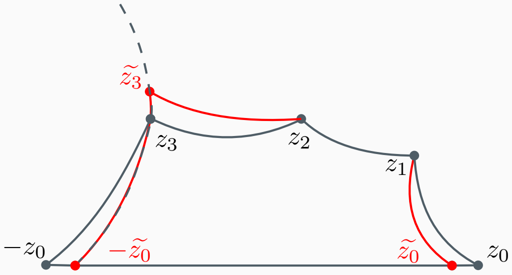
Solution : use their formulas but slightly modify the procedure

Generation of rational octagons

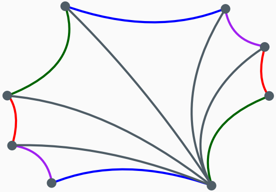
Solution : (1) replace z_0 by any $\widetilde{z}_0 \in \mathbb{Q}$ close to z_0



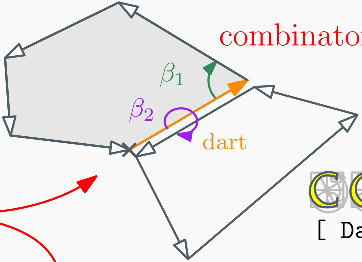
Generation of rational octagons



(2) replace z_3 by the unique $\widetilde{z}_3$
on the geodesic supporting $-\widetilde{z}_0$ and z_3
such that the pentagon
$\widetilde{z}_0, z_1, z_2, \widetilde{z}_3, -\widetilde{z}_0$ has area 2π $\implies \widetilde{z}_3 \in \mathbb{Q} + i\mathbb{Q}$

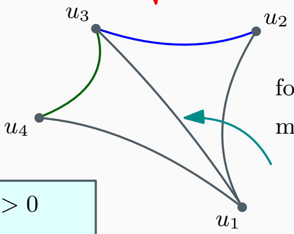


triangulation



combinatorial map …

[Damiand]



… and cross-ratios

forget points

maintain cross-ratios

$$[u_1, u_2, u_3, u_4] = \frac{(u_4 - u_2)(u_3 - u_1)}{(u_4 - u_1)(u_3 - u_2)}$$

$\texttt{Im}[u_1, u_2, u_3, u_4] > 0$
$\Leftrightarrow$
edge is Delaunay-flippable

4. Further prospects

- Towards higher genus
- Open questions

Problem

rational $4g$-gon G :

- opposite sides almost the same length
- total area almost $4\pi(g-1)$

$\downarrow$?

rational $4g$-gon G' nearby G :

- opposite sides exactly the same length
- total area exactly $4\pi(g-1)$

Towards higher genus

Problem

rational $4g$-gon G :

- opposite sides almost the same length
- total area almost $4\pi(g-1)$

?

~~rational~~ $4g$-gon G' nearby G :

- opposite sides exactly the same length
- total area exactly $4\pi(g-1)$

independent of genus

"small" algebraic extension :

$\mathbb{Q}[X]/P$ for some $P \in \mathbb{Q}[X]$
such that $\deg P$ is "small"

ex:
represent $\{Q(\sqrt{2}) \mid Q \in \mathbb{Q}[X]\}$
by $\mathbb{Q}[X]/(X^2-2)$

Open questions

Open questions

- Higher genus.
- Generate the whole mapping class group while working with rationals.
- Conjecture $O(\Delta \cdot n^2)$ for the Delaunay flip algorithm.

Thank you !