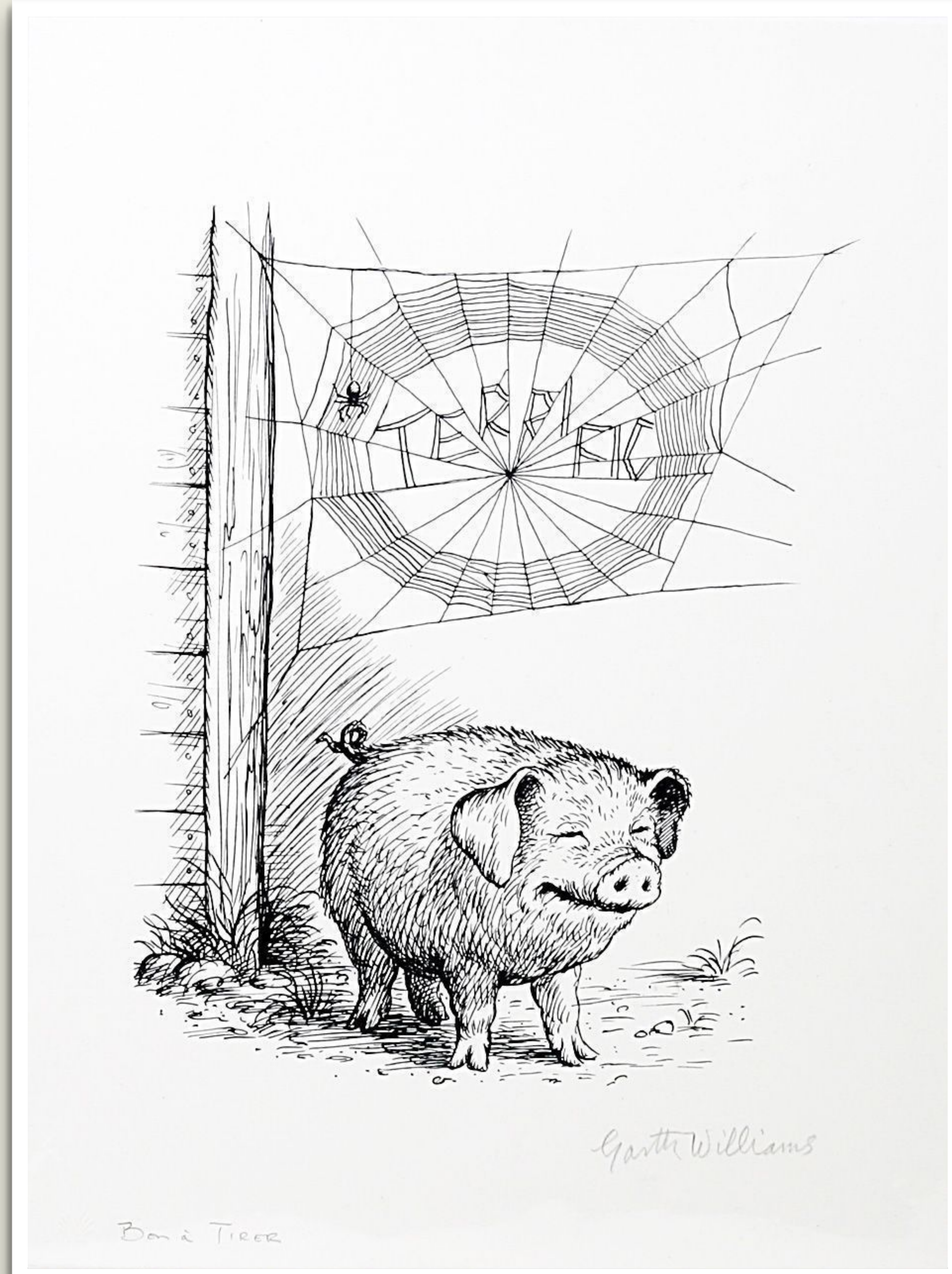
Fun with Toroidal Spring Embeddings!

Jeff Erickson
University of Illinois, Urbana-Champaign

CIRM workshop “Structures on Surfaces”
May 3, 2022

Joint work with Erin Chambers, Patrick Lin, and Salman Parsa



E. B. White, *Charlotte's Web* (1952)
Illustrated by Garth Williams

HOW TO DRAW A GRAPH

By W. T. TUTTE

[Received 22 May 1962]

1. Introduction

WE use the definitions of (**11**). However, in deference to some recent attempts to unify the terminology of graph theory we replace the term 'circuit' by 'polygon', and 'degree' by 'valency'.

A graph G is 3-*connected* (*nodally* 3-*connected*) if it is simple and non-separable and satisfies the following condition; if G is the union of two proper subgraphs H and K such that $H \cap K$ consists solely of two vertices u and v, then one of H and K is a link-graph (arc-graph) with ends u and v.

It should be noted that the union of two proper subgraphs H and K of G can be the whole of G only if each of H and K includes at least one edge or vertex not belonging to the other. In this paper we are concerned mainly with nodally 3-connected graphs, but a specialization to 3-connected

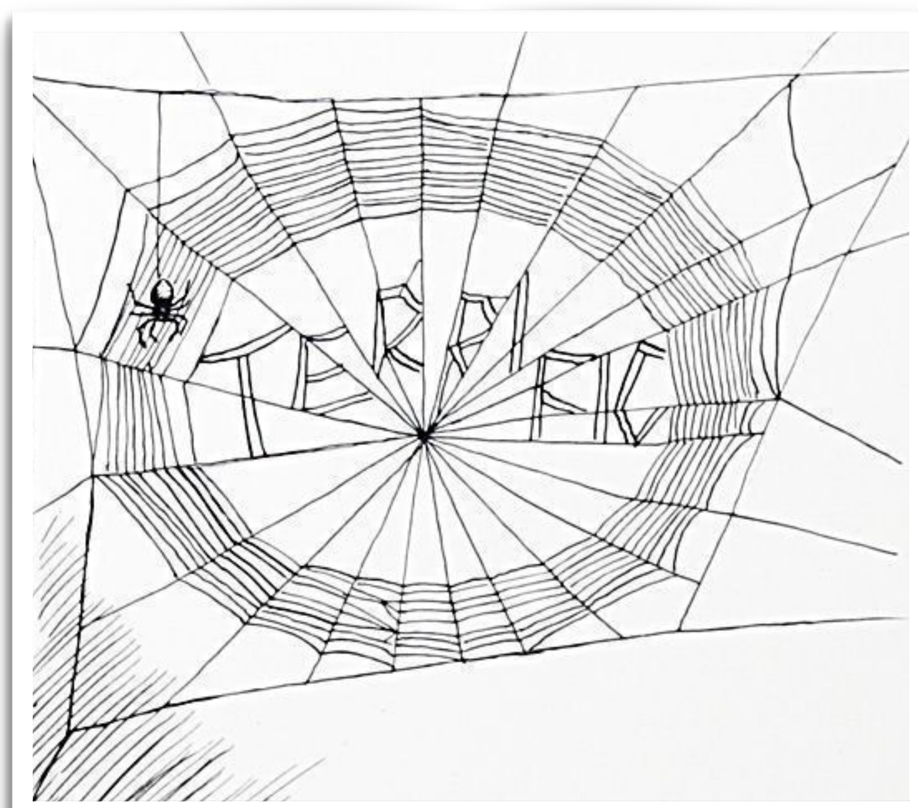
Spring embedding theorem

[Tutte 1963]

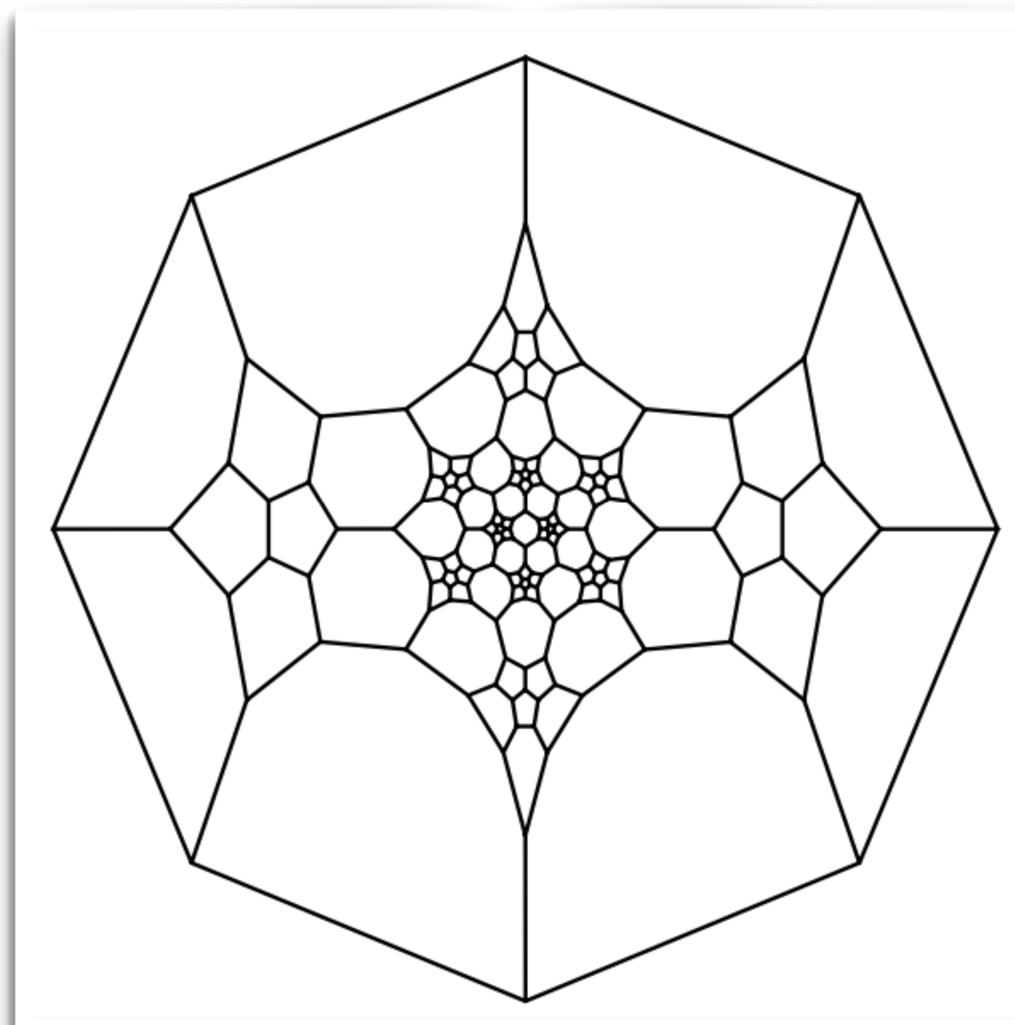
- Let G be a simple 3-connected planar graph, with arbitrary positive edge weights.
- Let Γ be a planar embedding of G whose outer face is a convex polygon.
- There is a unique embedding $\Gamma_=$ of G, equivalent to Γ and with the same outer face as Γ, such that every interior vertex is the weighted average of its neighbors.
- Every face of $\Gamma_=$ is convex.

Spring embedding theorem

Think of edges as springs or rubber bands.
Let the system relax to equilibrium.



[Williams 52]



[Delgado-Friedrichs 03]

Spring embedding algorithm

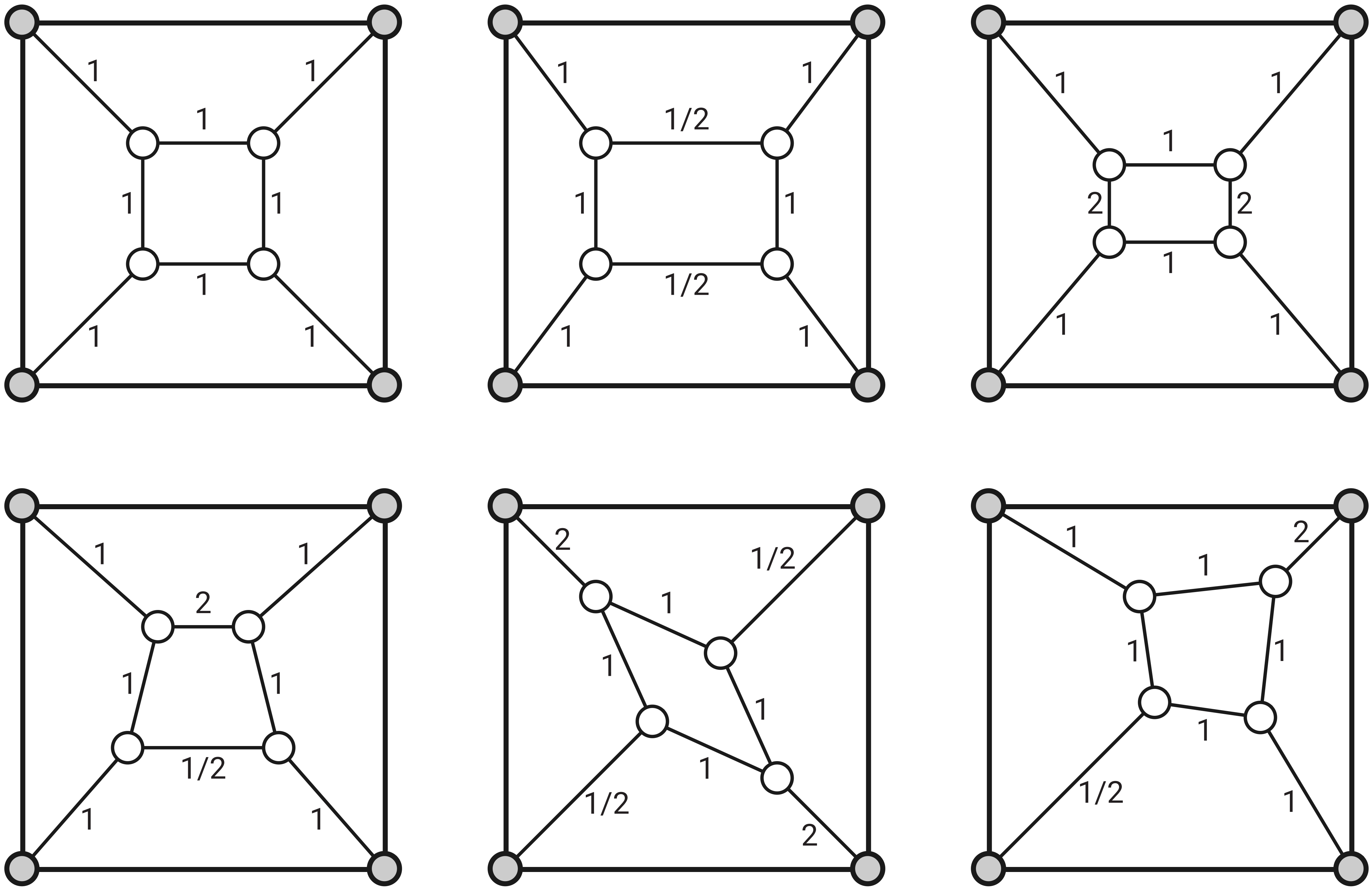
[Tutte 1963]

- Minimize potential energy

$$\Phi := \sum_{e} \omega_e \cdot |e|^2$$

- Solve linear system $\nabla\Phi = 0$:
  For every interior vertex *u*:

$$\sum_{v} \omega_{uv}(x_v - x_u) = 0$$
$$\sum_{v} \omega_{uv}(y_v - y_u) = 0$$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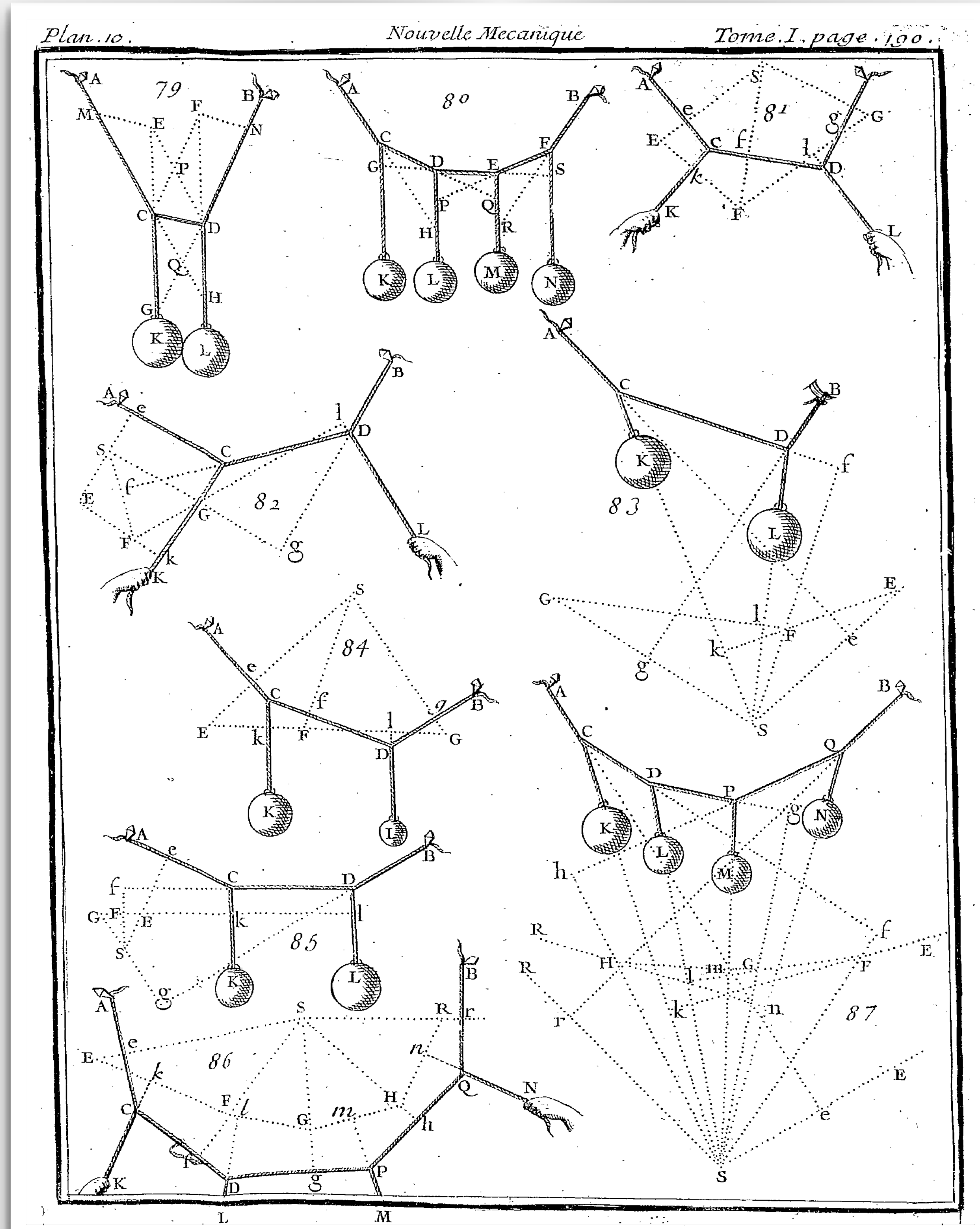
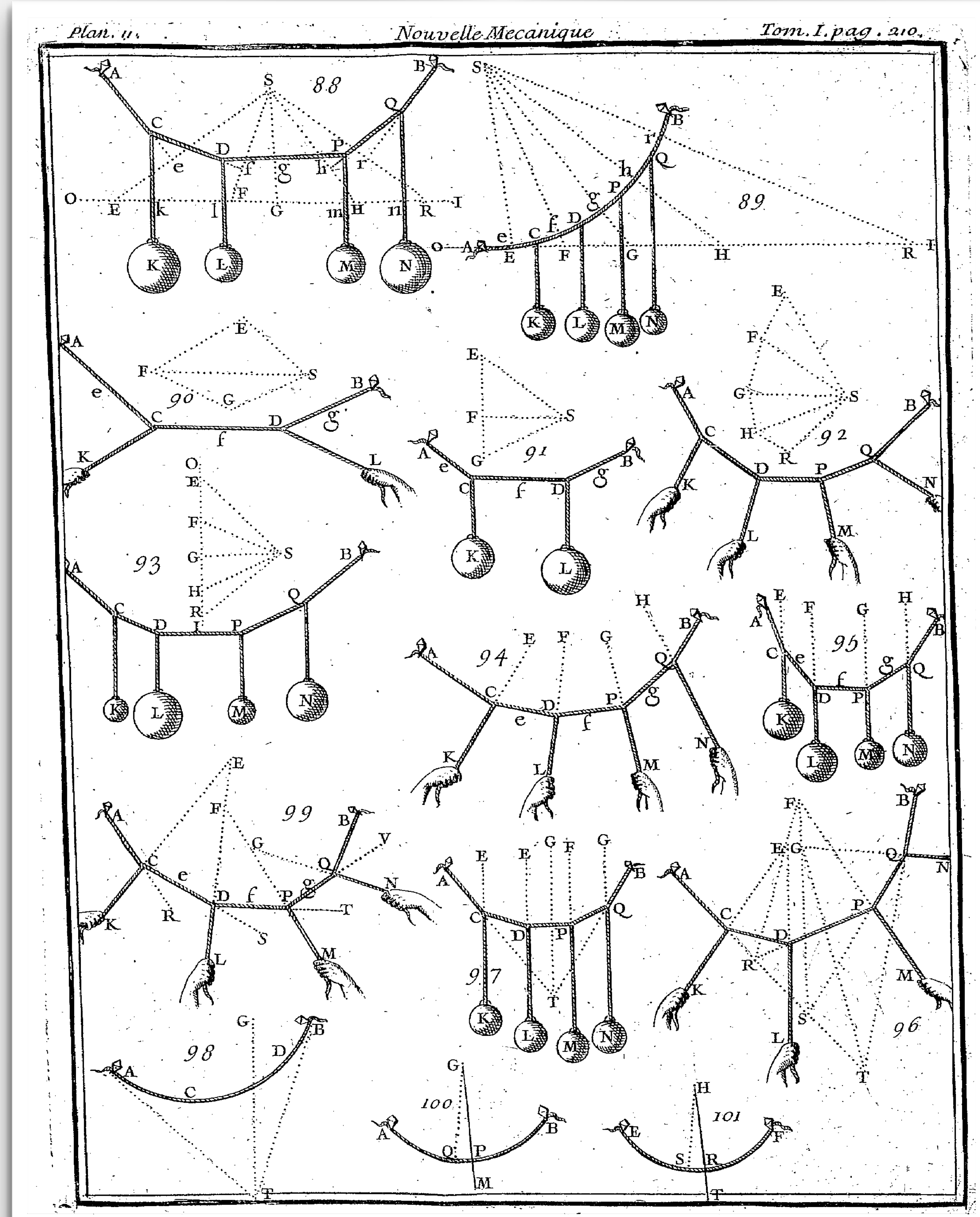
13. Unsolved problems

The result of § 12 raises the following questions. Can we construct simultaneous straight representations, with intersections limited as above, of G and G^* in which the residual regions of each representation are convex? Or such that corresponding edges are represented by perpendicular segments?

Finally we may remark that very little is known about representations of graphs in the projective plane and higher surfaces (**4**).

[Tutte 63]

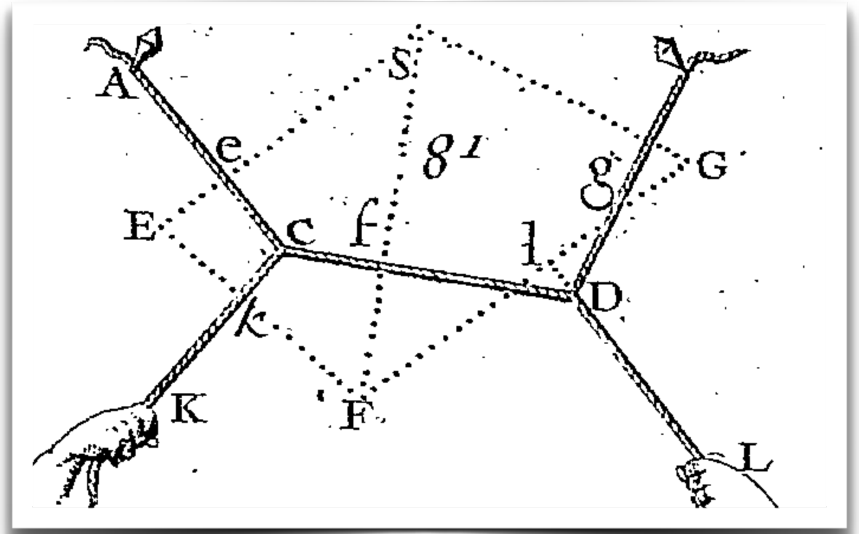




[Varignon 1725]

Pierre Varignon

Nouvelle mécanique, ou statique, dont le projet fut donné en MDCLXXXVII (1725)



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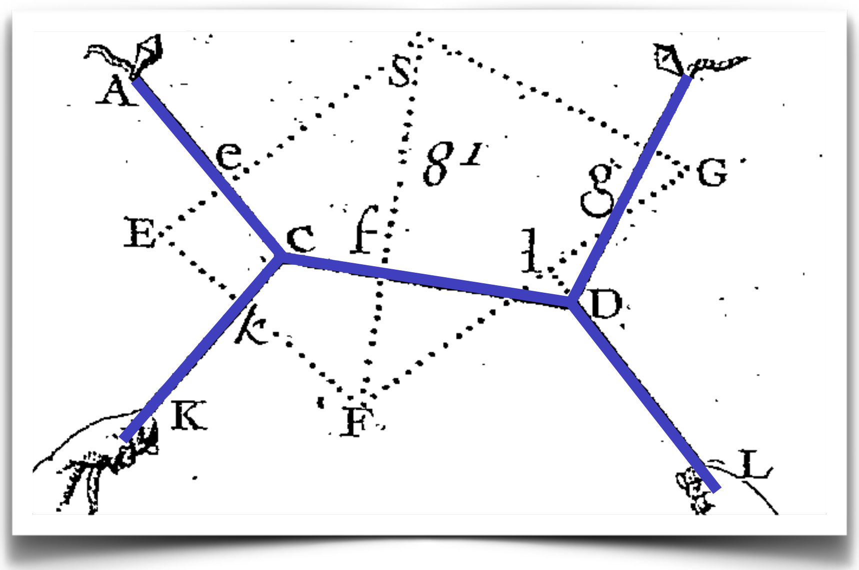

Primal: Form diagram (“funicular polygon”)

Pierre Varignon

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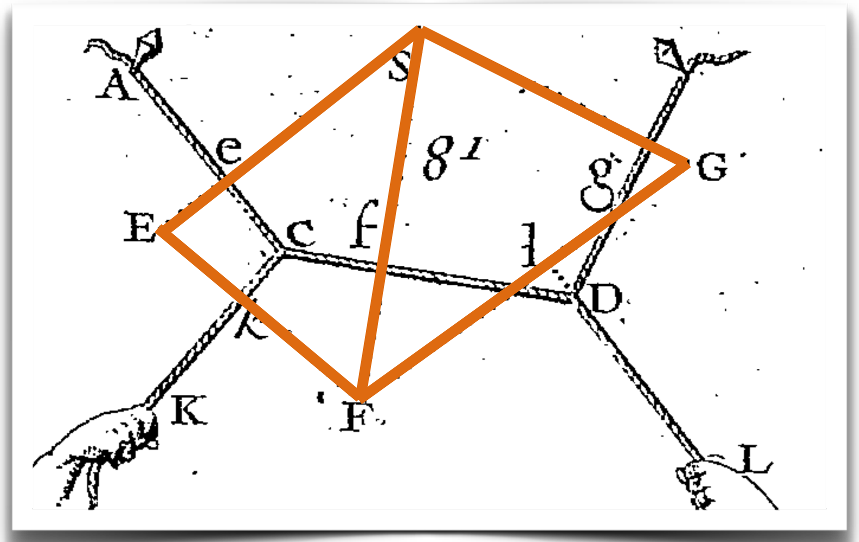
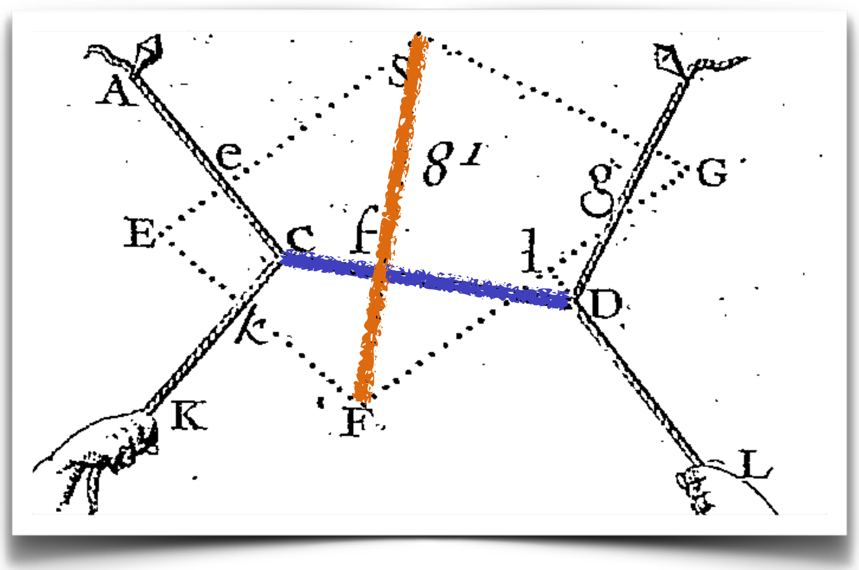


Primal: Form diagram (“funicular polygon”)

Dual: Force diagram (“force polygon”)

Pierre Varignon

Nouvelle mécanique, ou statique, dont le projet fut donné en MDCLXXXVII (1725)



Primal: Form diagram ("funicular polygon")

Dual: Force diagram ("force polygon")

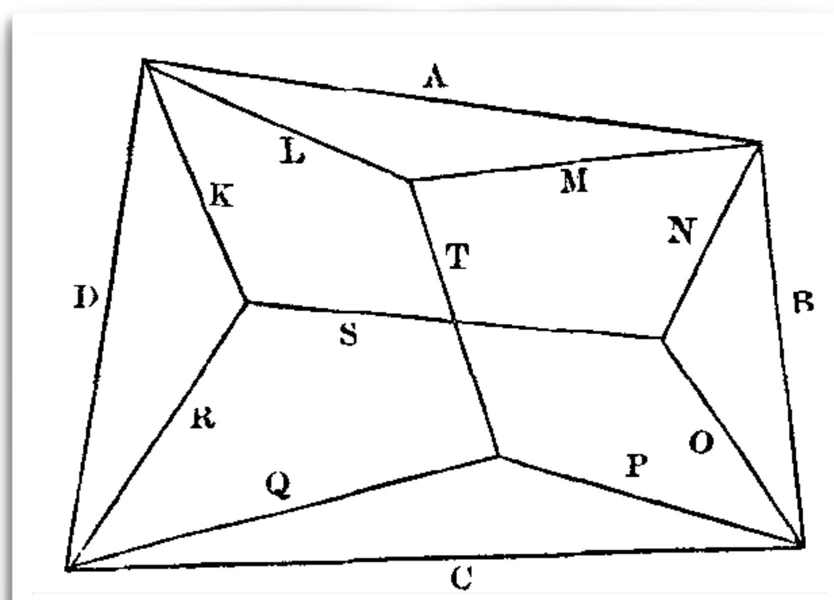
Corresponding edges are perpendicular

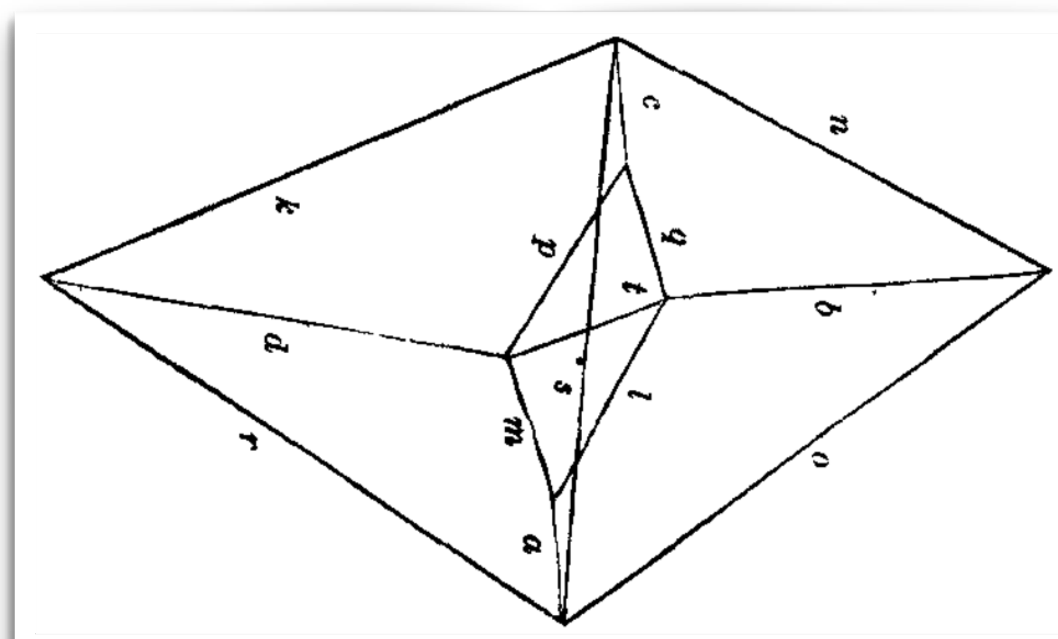
(for this talk)

James Clerk Maxwell

On Reciprocal Figures, Frames, and Diagrams of Forces (1870)

Definition.—Two plane rectilinear figures are reciprocal when they consist of an equal number of straight lines, so that corresponding lines in the two figures are at right angles, and corresponding lines which meet in a point in the one figure form a closed polygon in the other.





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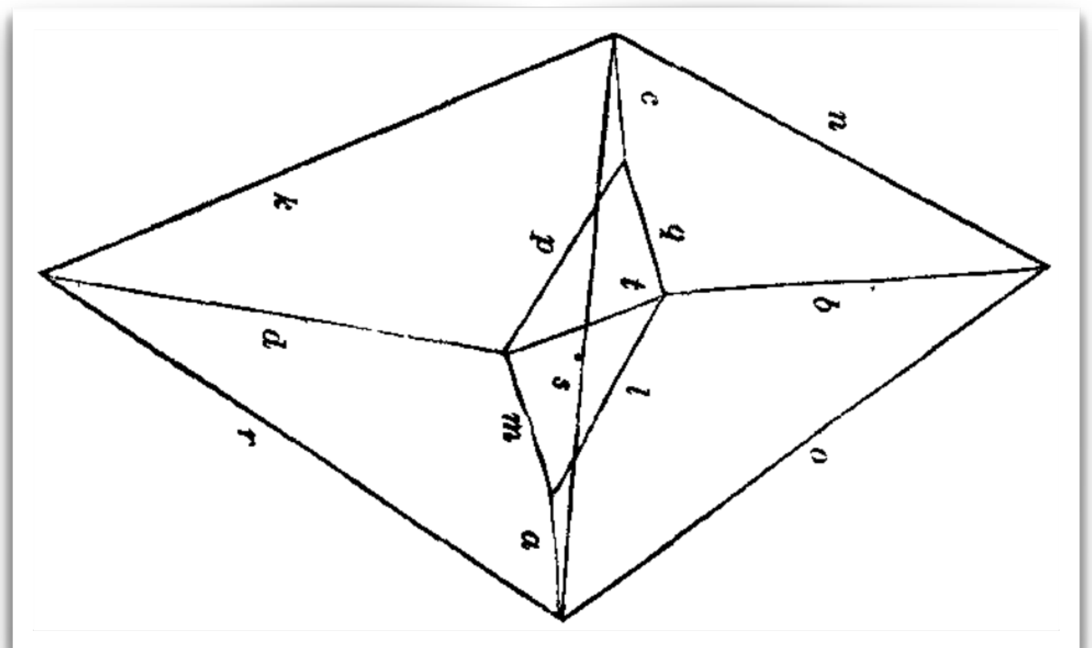
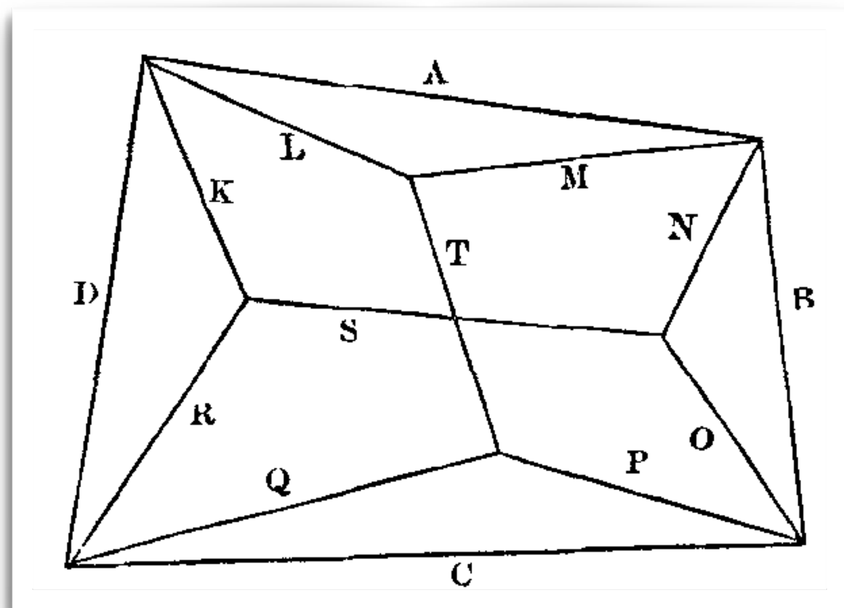




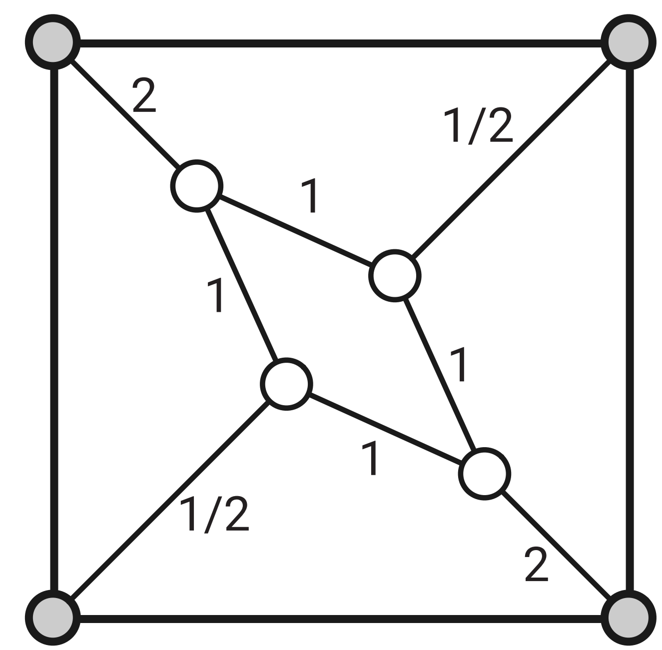
Equilibrium stress

[Maxwell 1864, 1870]

- Fix a straight-line plane embedding Γ with a convex outer face
- Assign a ***stress ω_e>0*** to every internal edge e
- ω is an ***equilibrium*** stress iff every interior vertex is the weighted average of its neighbors:

$$\sum_v \omega_{uv}(x_u - x_v) = 0$$

$$\sum_v \omega_{uv}(y_u - y_v) = 0$$

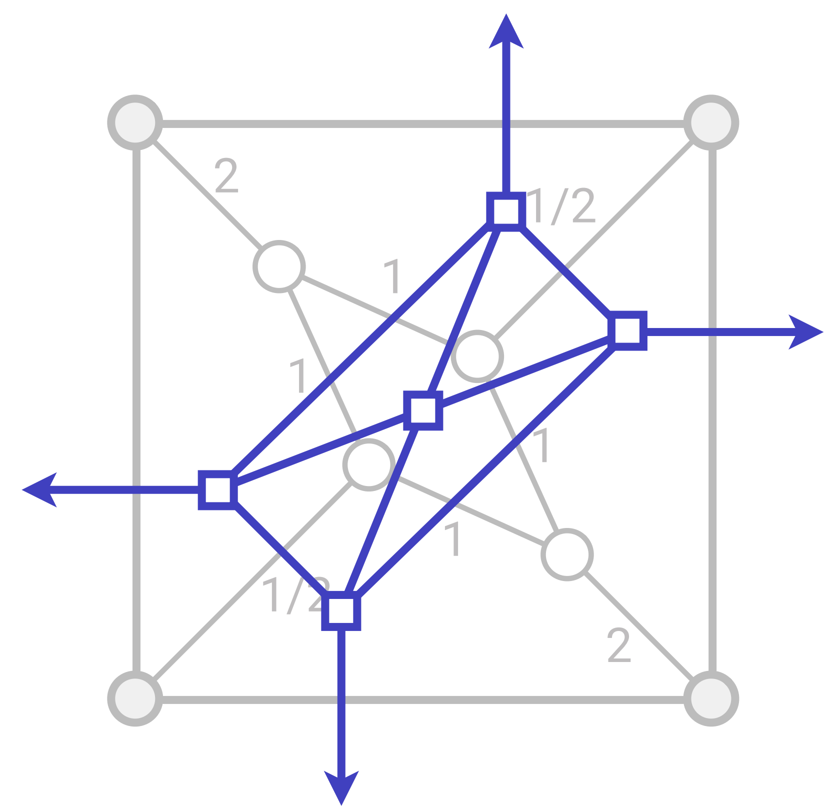


Maxwell–Cremona correspondence

[Maxwell 1864, 1870]
[Whiteley 1982]
[Crapo Whiteley 1983]

- Every equilibrium stress for Γ defines a *reciprocal diagram* Γ^* and vice versa.
- Straight-line embedding dual to Γ

$$\boxed{\begin{gathered} e^* \perp e \\ |e^*| = \omega_e \cdot |e| \end{gathered}}$$



- Faces of Γ^* certify equilibrium at vertices of Γ

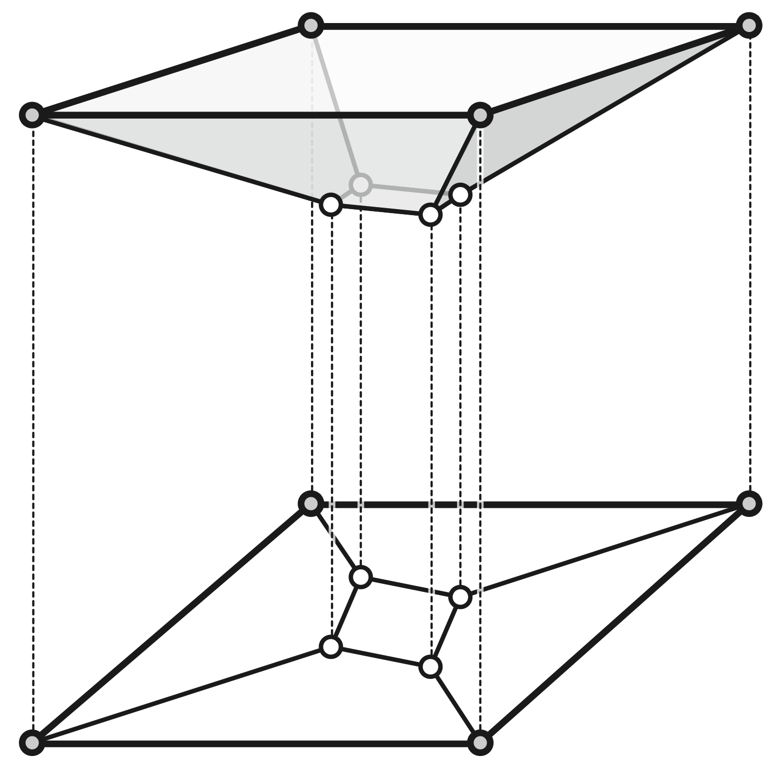
Maxwell–Cremona correspondence

[Maxwell 1864, 1870]

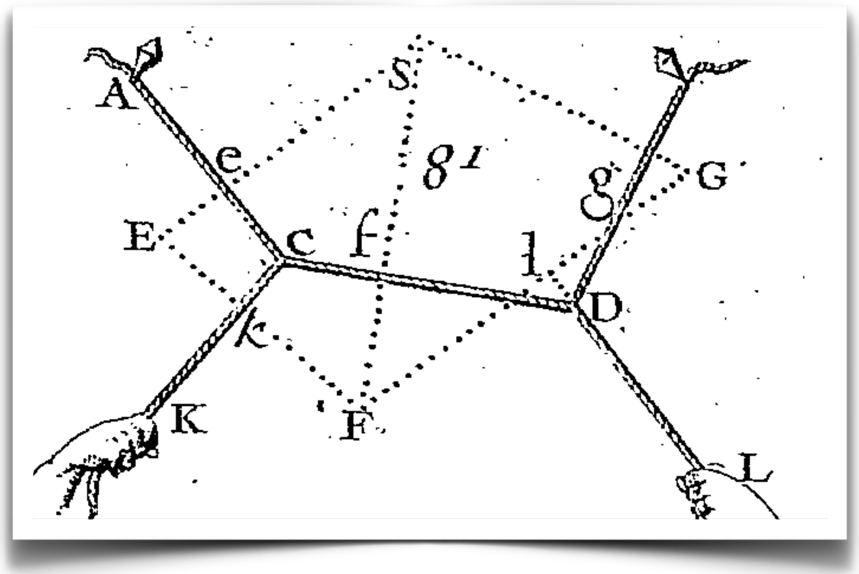
[Whiteley 1982]
[Crapo Whiteley 1983]

- Every equilibrium stress for Γ defines a *convex polyhedral lifting* $\Gamma^{\uparrow}$ and vice versa.

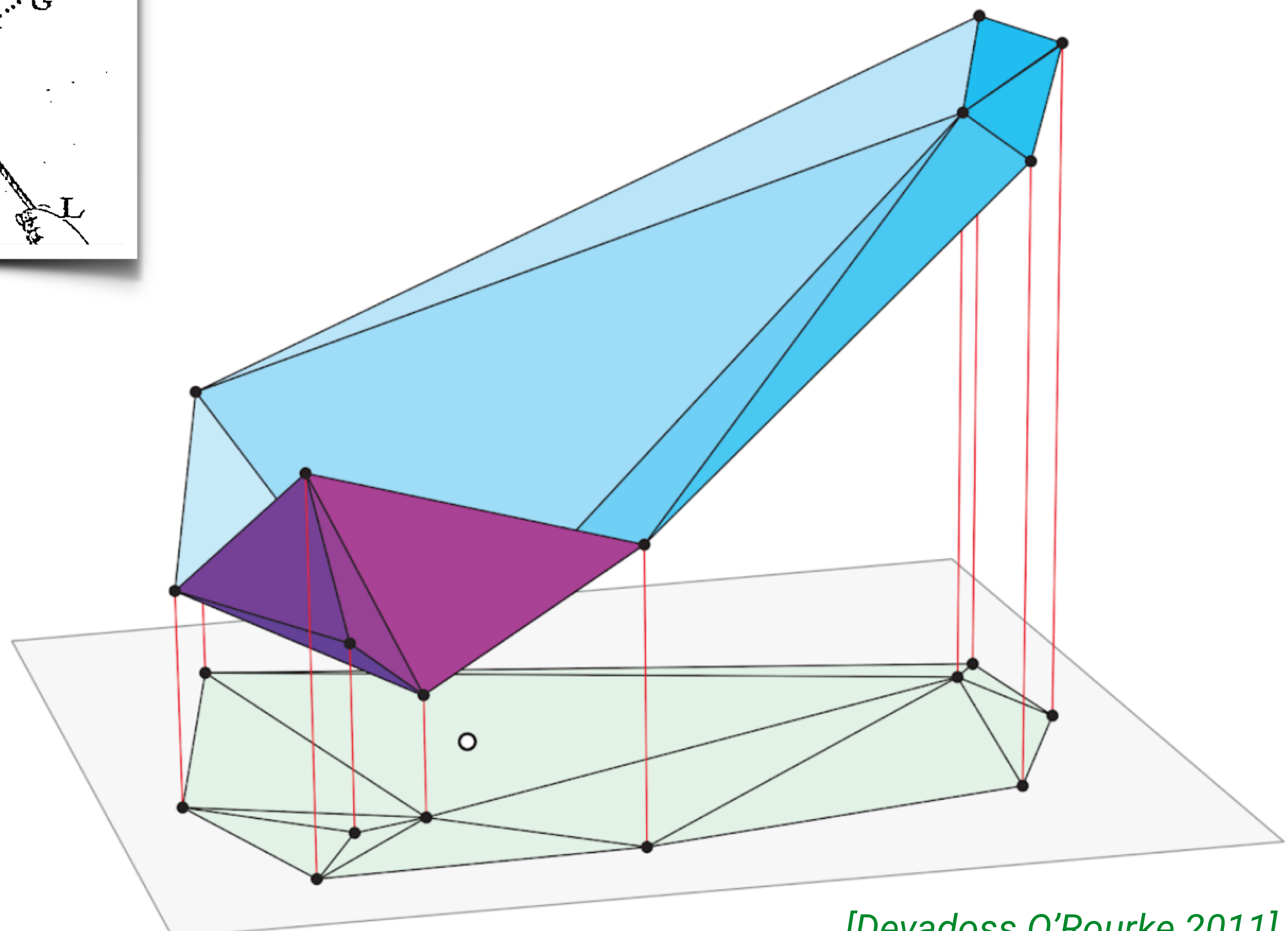
- $\Gamma^{\uparrow}$ is a straight-line graph in 3-space
  - Γ is the orthogonal projection of $\Gamma^{\uparrow}$
  - $\Gamma^{\uparrow}$ is not coplanar
  - Each interior face f lifts to a planar polygon $f^{\uparrow}$
  - Each interior edge e lifts of a convex edge $e^{\uparrow}$



[Steinitz 1916]



[Varignon 1725]



[Devadoss O'Rourke 2011]

Delaunay/Voronoi lifting

[Voronoï 1908] [Brown 1980] [Seidel 1982] [Edelsbrunner Seidel 1985]

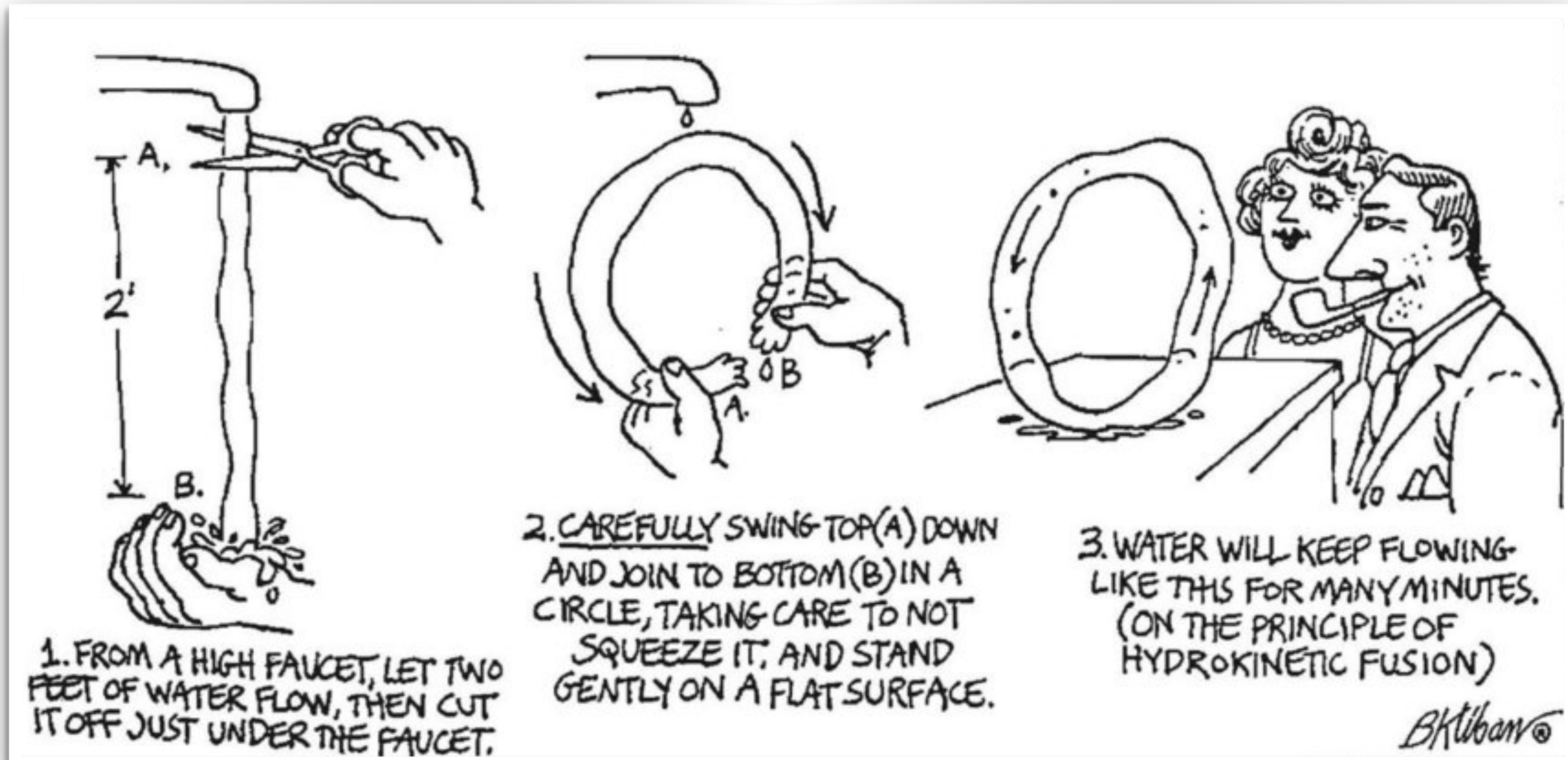
- For any weighted point $p = ((a, b), \pi)$ in the plane, define
  - Lifted point $p^{\uparrow} = (a, b, \frac{1}{2}(a^2+b^2) - \pi)$
  - Dual plane p^*: $z = ax + by - \frac{1}{2}(a^2+b^2) + \pi$
- *Delaunay(P)* = projection of lower convex hull of $P^{\uparrow}$
  - "regular / coherent subdivision"
- *Voronoi(P)* = projection of upper envelope of P^*
  - "power / Laguerre diagram"

Maxwell–Cremona–*Delaunay* correspondence

For any planar straight-line graph Γ with a convex outer face, the following are (essentially) equivalent:

- *Positive* equilibrium stress ω for Γ
- *Embedded* reciprocal diagram Γ^*
- *Convex* polyhedral lifting $\Gamma^\uparrow$
- *Delaunay* vertex weights for Γ

Let’s add some topology!



B. Kliban, *Advanced Cartooning and Other Drawings* (1993)

The flat torus

- Identify opposite sides of any parallelogram

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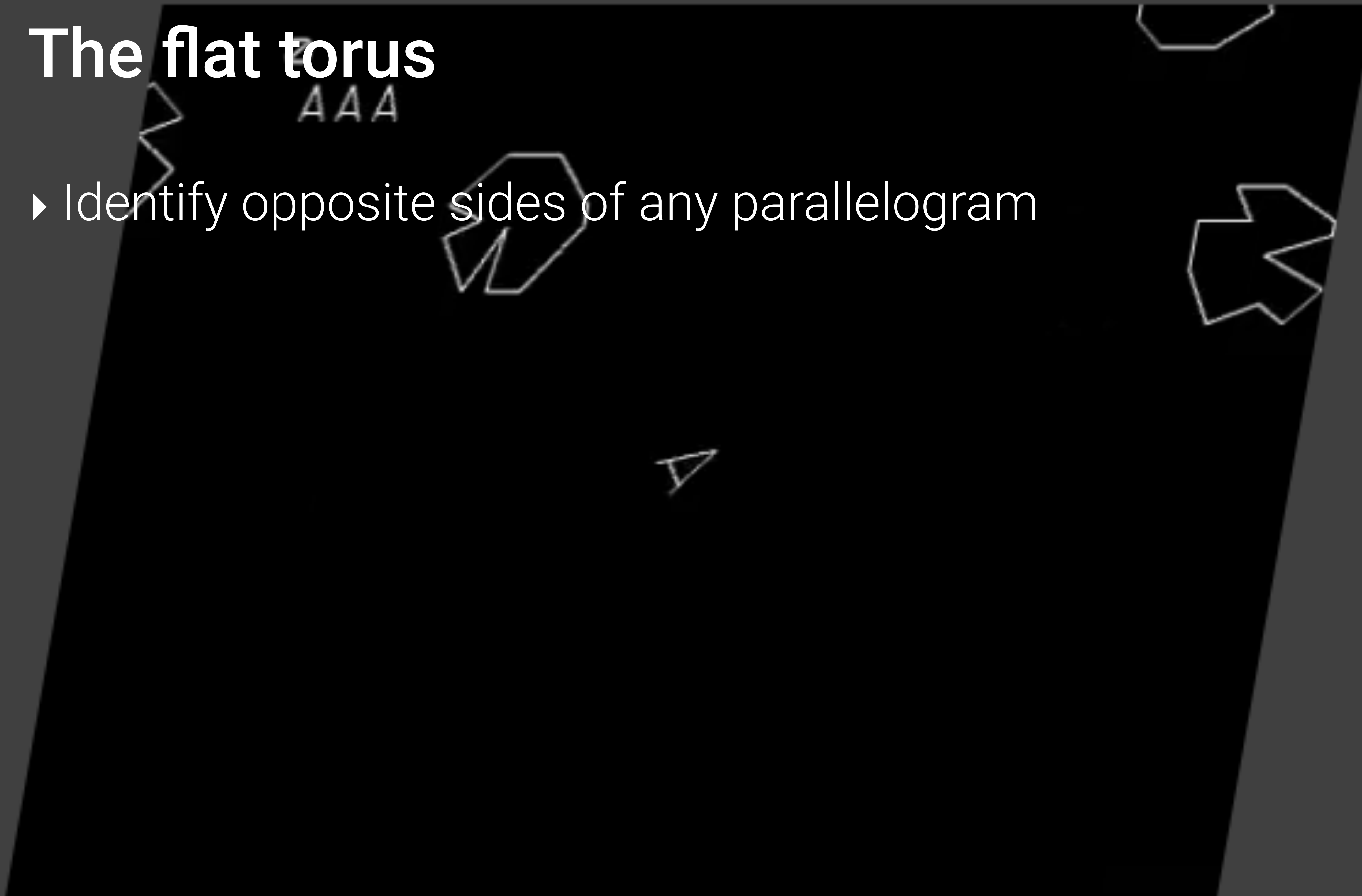
- Identify opposite sides of any parallelogram



The flat torus

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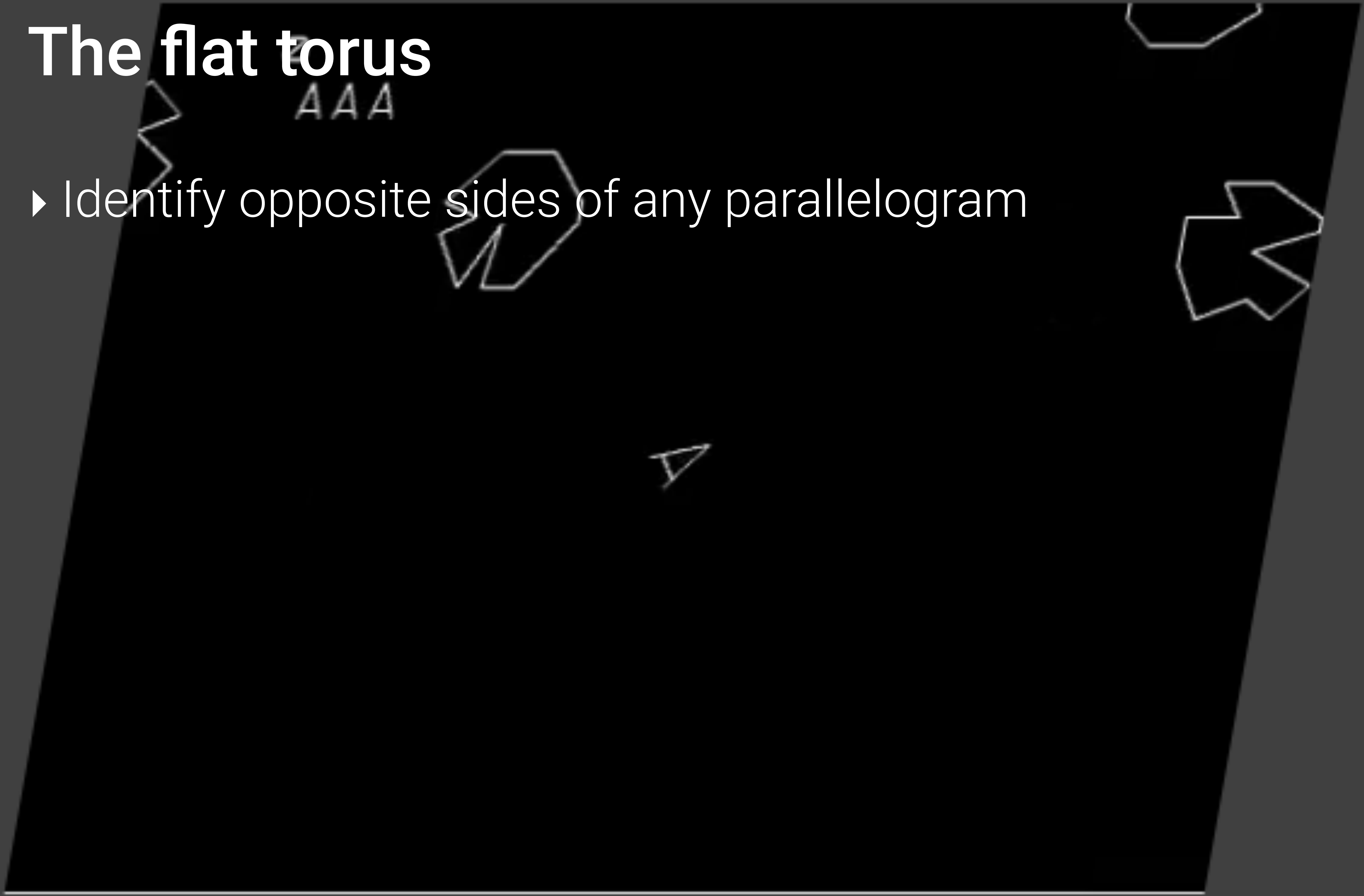
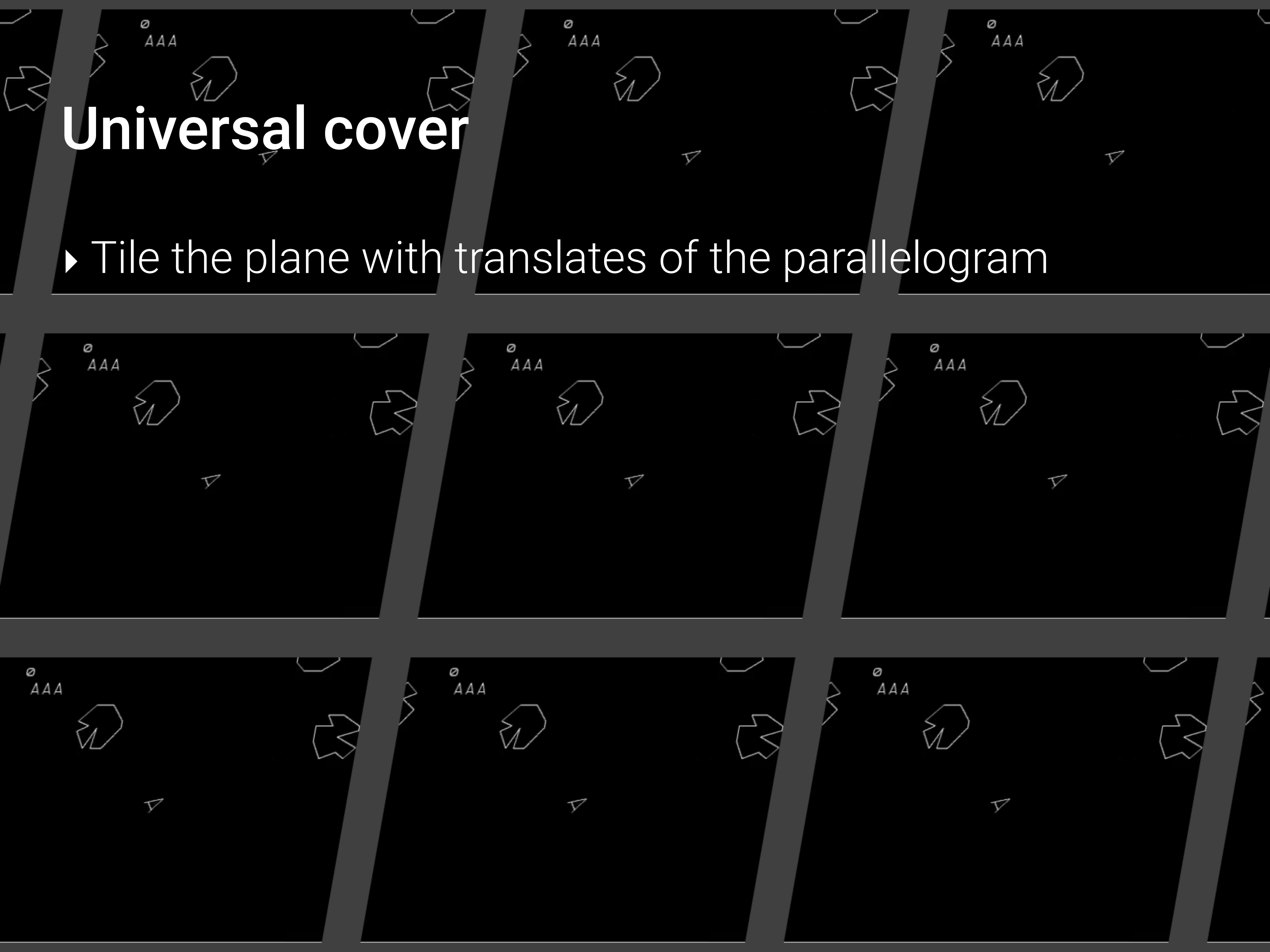


Universal cover

- Tile the plane with translates of the parallelogram



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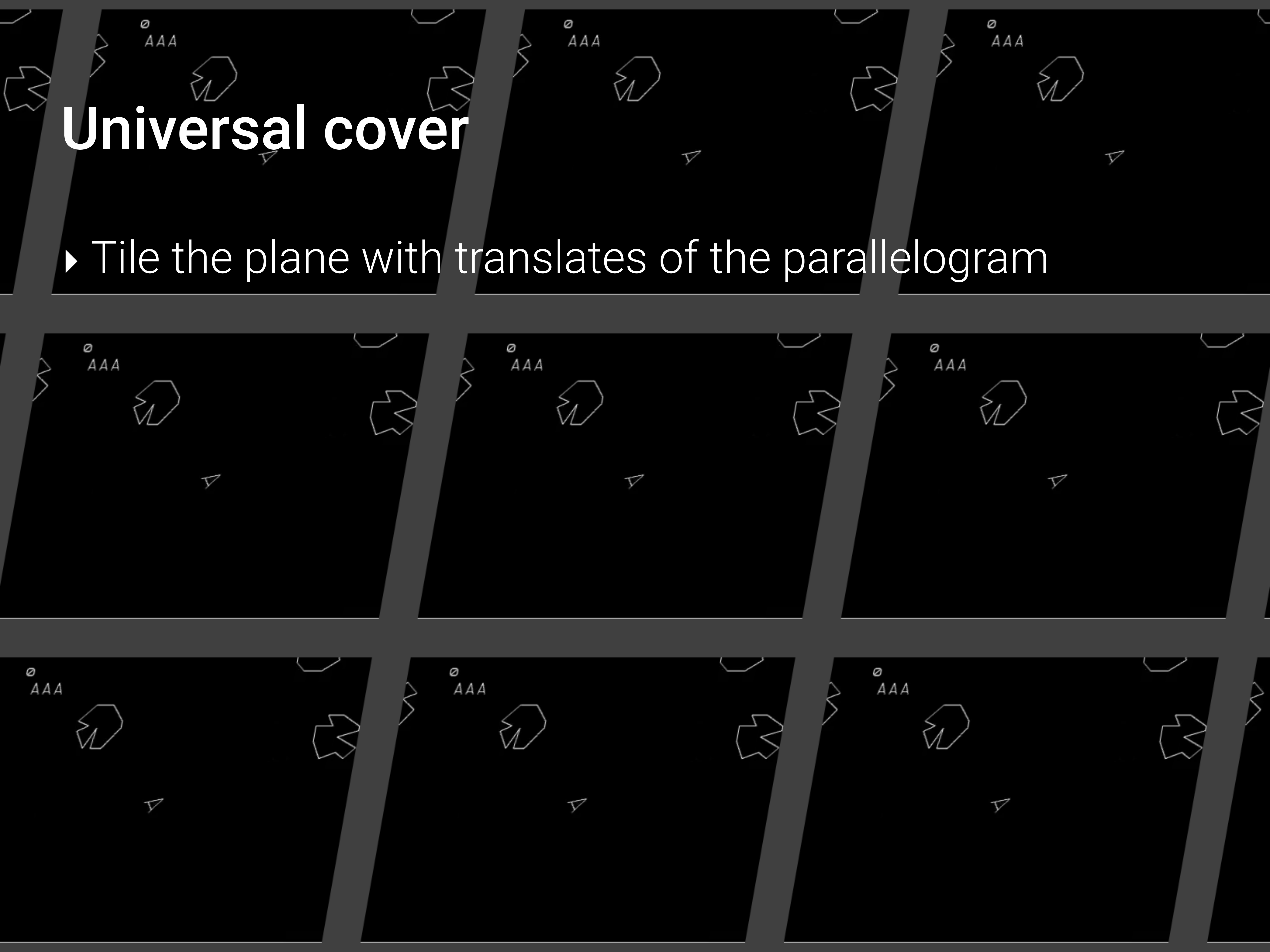
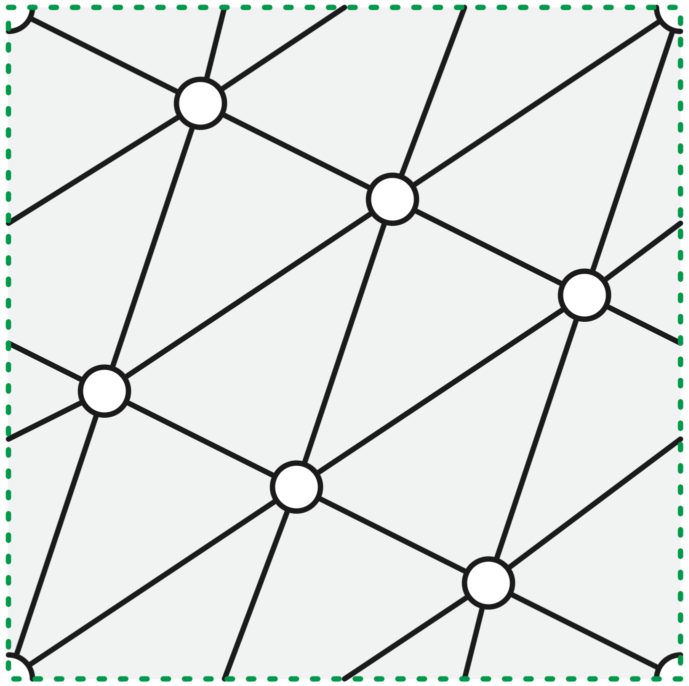
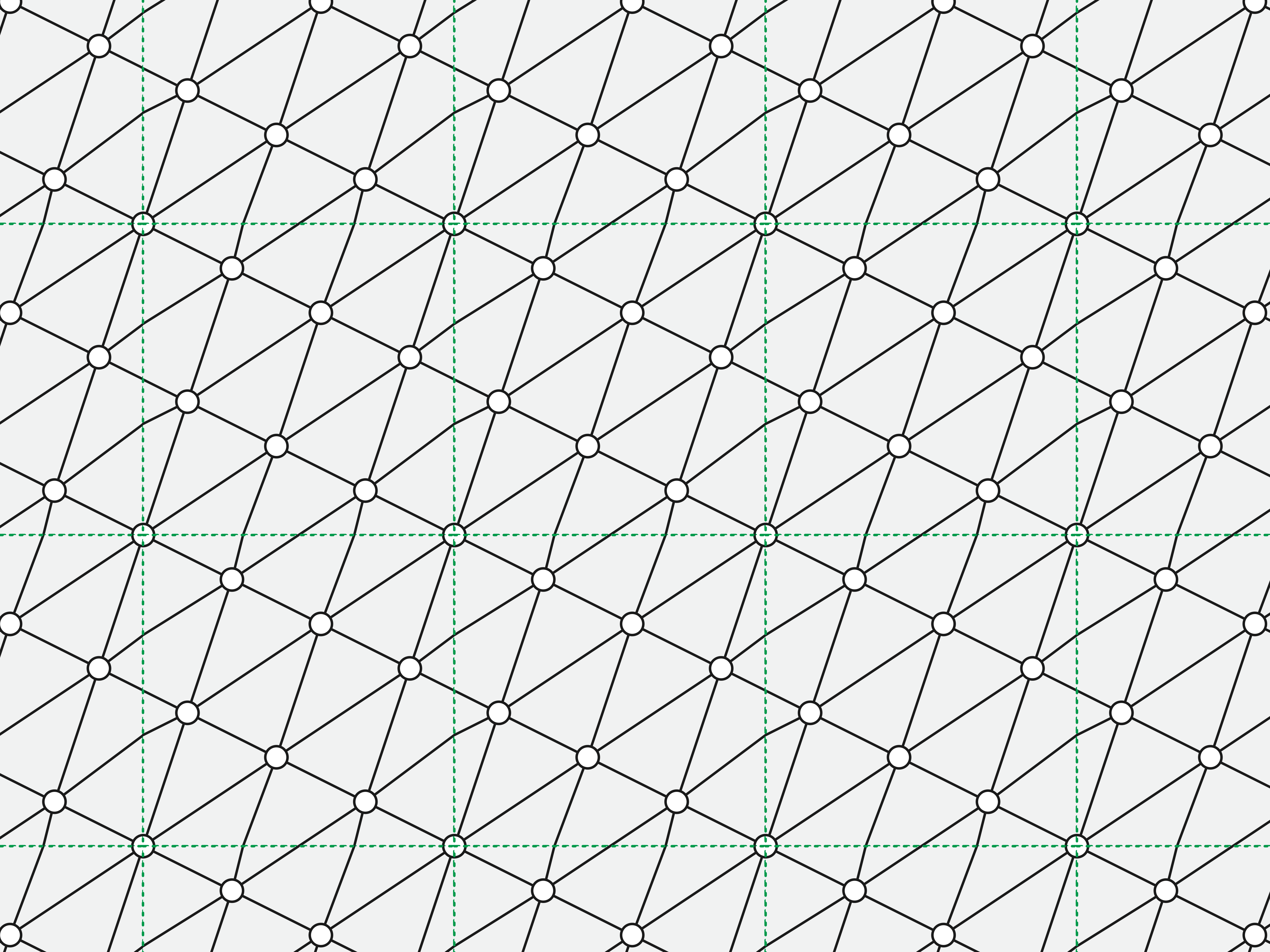


Geodesic embeddings

- *Geodesic* = projection of a line segment in the universal cover
- Geodesic *embedding* = projection of an infinite periodic straight-line plane graph in the universal cover





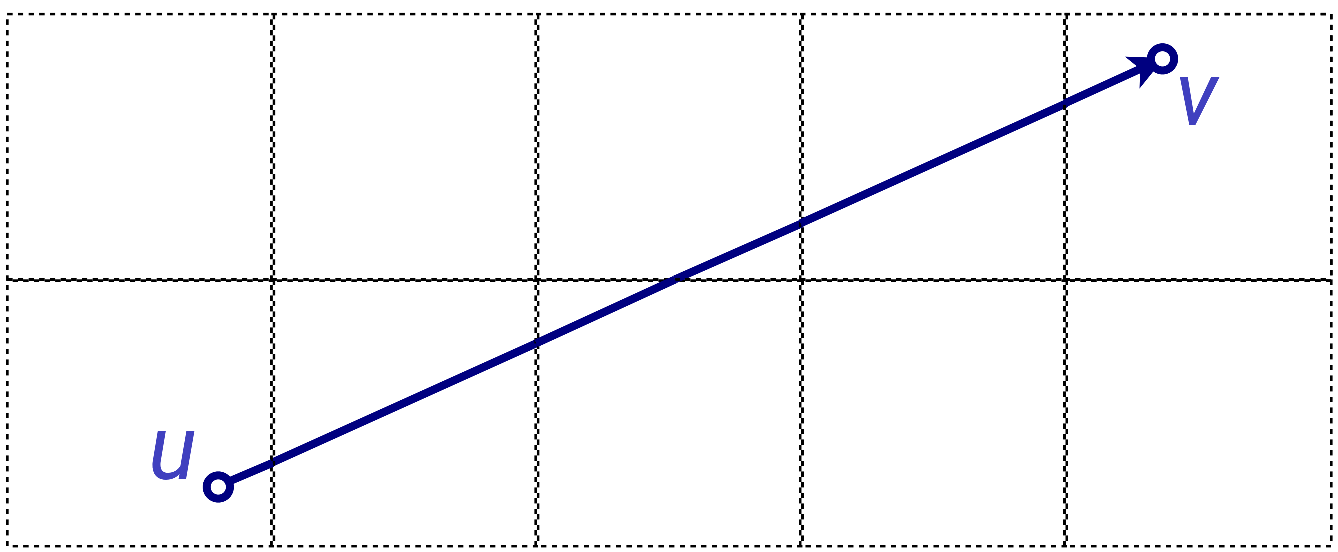
Geodesic embeddings

[Karp Miller Winograd 67] [Waite 67] [Collatz 78]
[Iwano Steiglitz 78] [Kosaraju Sullivan 88]
[Rao Kalath 88] [Orlin 84] [E Whittlesey 05]
[Borcea Streinu 10, 15] [Ross 11, 12, 12′]
[Tanagawa 12] [Malestein Theran 13]
[Kaszanitzky Schulze Tanigawa 19]

Any geodesic embedding on the flat torus can be represented by

- *Position* vector $p_v \in [0,1)^2$ for each vertex v
- *Translation* vector $\tau_{u\to v} \in \mathbb{Z}^2$ for each dart $u \to v$

$\tau_{u\to v} = (4, 1)$ $\tau_{v\to u} = (-4, -1)$



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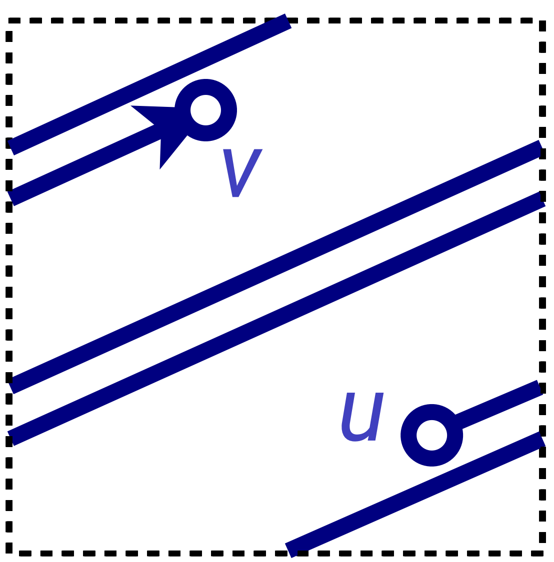

Spring embedding theorem!

[Yves Colin de Verdière 1990]
[Delgado-Friedrichs 2004]
[Steiner Fischer 2004] [Lovász 2004]
[Gortler Gotsman Thurston 2006]
[Hass Scott 2015]

- Let Γ be any ***essentially simple, essentially 3-connected*** embedding of a graph on any flat torus, with arbitrary positive edge weights.

- There is an ***essentially unique*** geodesic embedding $\Gamma_=$ ***isotopic*** to Γ where every vertex is in weighted equilibrium with respect to its neighbors. Every face of $\Gamma_=$ is convex.

Spring embedding theorem!

[Yves Colin de Verdière 1990]
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- Let Γ be any *essentially simple* *essentially 3-connected* embedding of a graph on any flat torus, with arbitrary positive edge we

Universal cover is simple
(property of embedding, not G)

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Spring embedding theorem!

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Universal cover is 3-connected
(property of embedding, not G)

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Minimal combinatorial requirements

- There is an *essentially unique* geodesic embedding $\Gamma_=$ *isotopic* to Γ where every vertex is in weighted equilibrium with respect to its neighbors. Every face of $\Gamma_=$ is convex.

[Yves Colin de Verdière 1990]
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Spring embedding theorem!

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- There is an *essentially unique* geodesic embedding $\Gamma_=$ *isotopic* to Γ where every vertex is in weighted equilibrium with respect to its ne s convex.

Unique up to translation

No fixed vertices!

Spring embedding theorem!

[Yves Colin de Verdière 1990]
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- There is an ***essentially unique*** geodesic embedding $\Gamma_=$ *isotopic* to Γ where every vertex is in weighted equilibrium with respect to its neighbors.

> Reachable by continuously deforming the surface
>
> = combinatorially ***and homologically*** equivalent

Spring embedding theorem!

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Spring embedding algorithm

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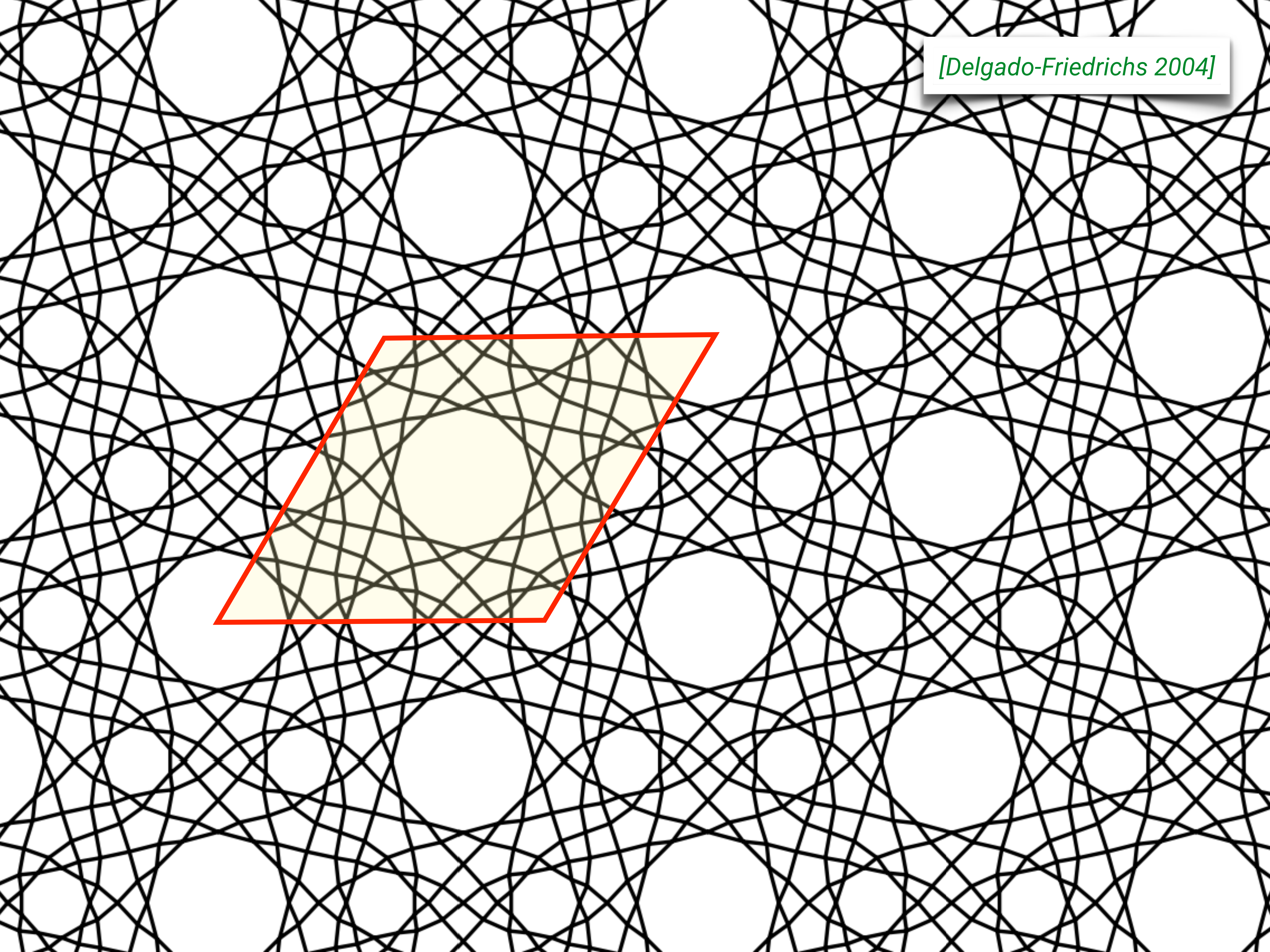
- Minimize potential energy

$$\Phi := \sum_e \omega_e \cdot |e|^2$$

- Solve linear system $\nabla\Phi = 0$ for vertex positions p_v

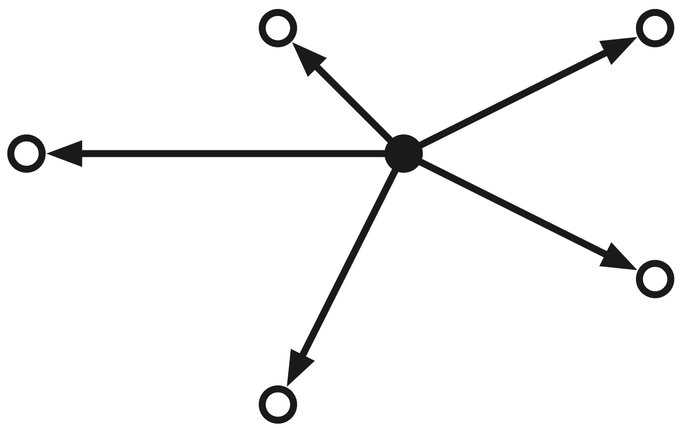
  For every vertex u:

$$\sum_v \omega_{uv}(p_v - p_u + \tau_{u\to v}) = (0,0)$$

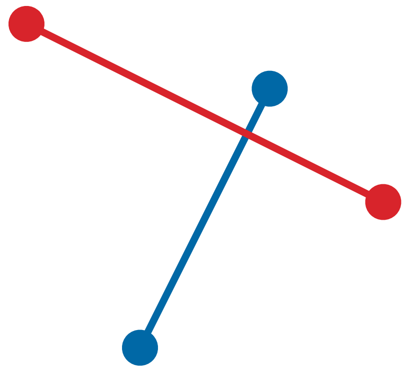


[Delgado-Friedrichs 2004]

Local metric properties

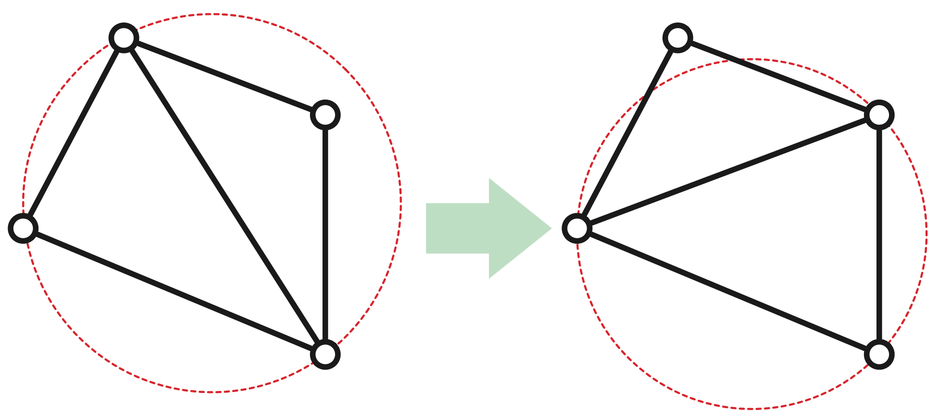


Equilibrium



Orthogonal

Delaunay

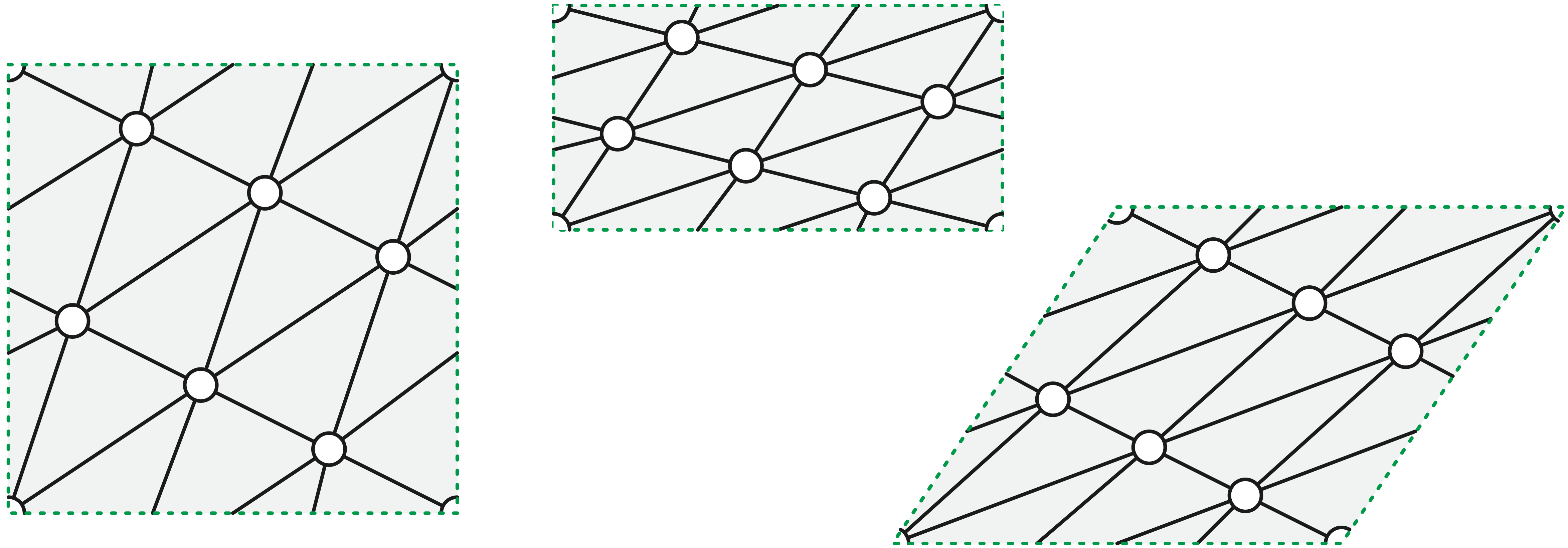


[Voronoi 1908] [Bobenko Springborn 07]

Equilibrium is shape-agnostic

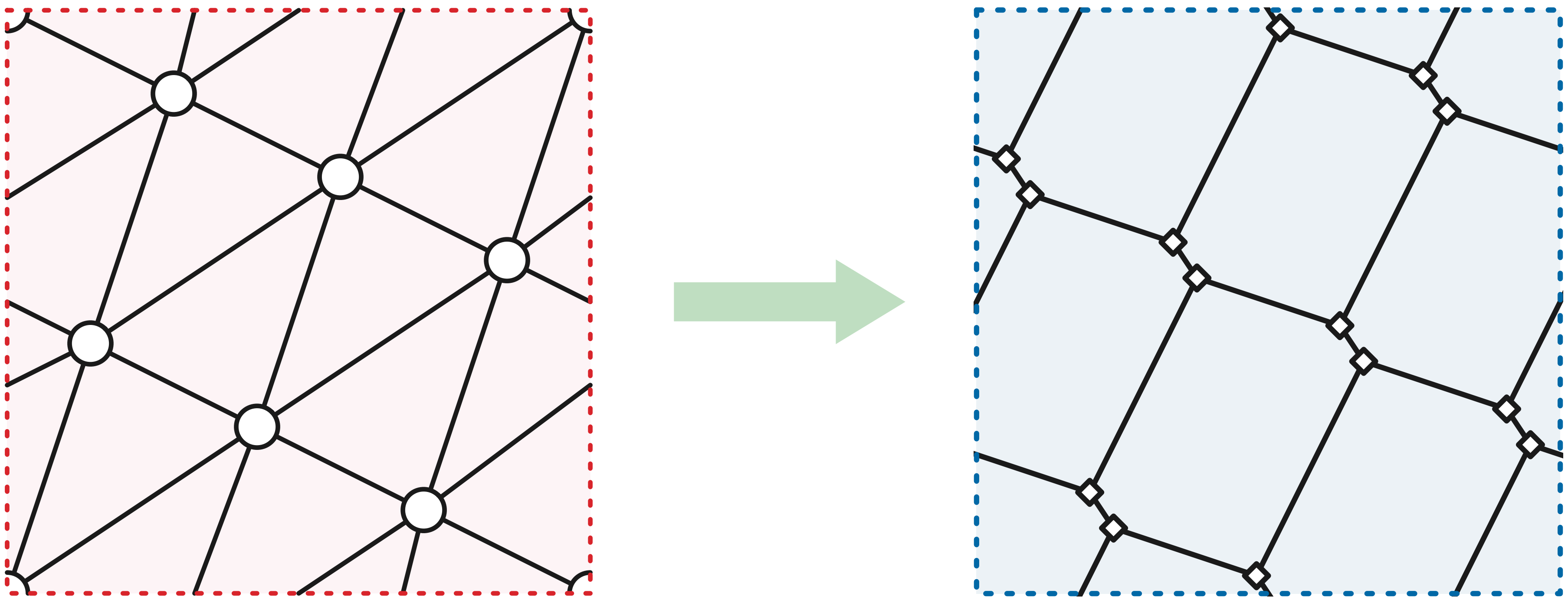
[Delgado-Friedrichs 2004]

If ω is an equilibrium stress for Γ on *any* flat torus,
then ω is an equilibrium stress for
the *image* of Γ on *every* flat torus.



Reciprocal diagram

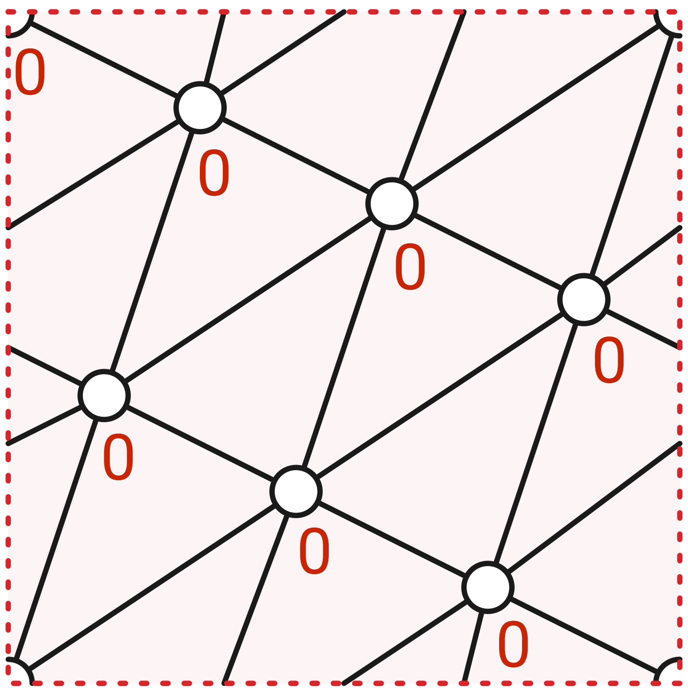
Geodesic embedding of Γ^* on *the same* flat torus as Γ, where every edge e is orthogonal to its dual edge e^*.



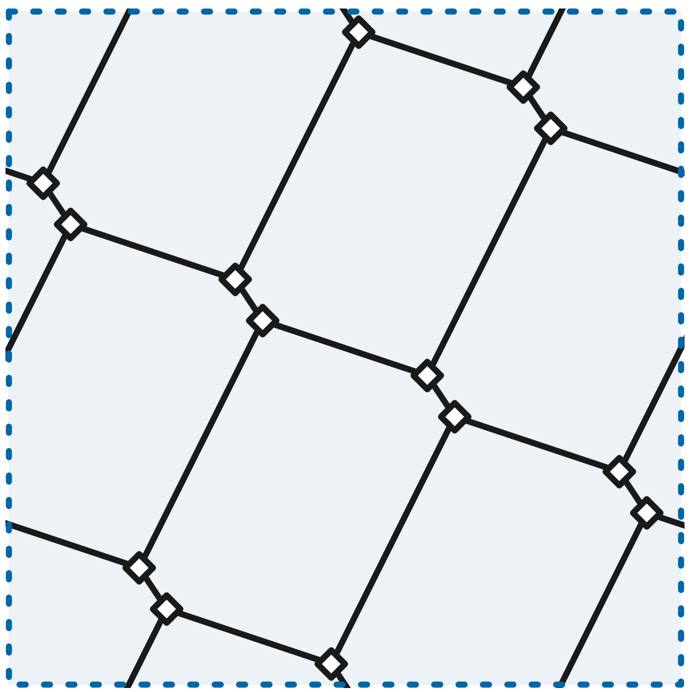
Delaunay ⇔ reciprocal

[E Lin 2020]

Any vertex weights that make Γ Delaunay
define a reciprocal diagram Γ^* and vice versa.



Delaunay

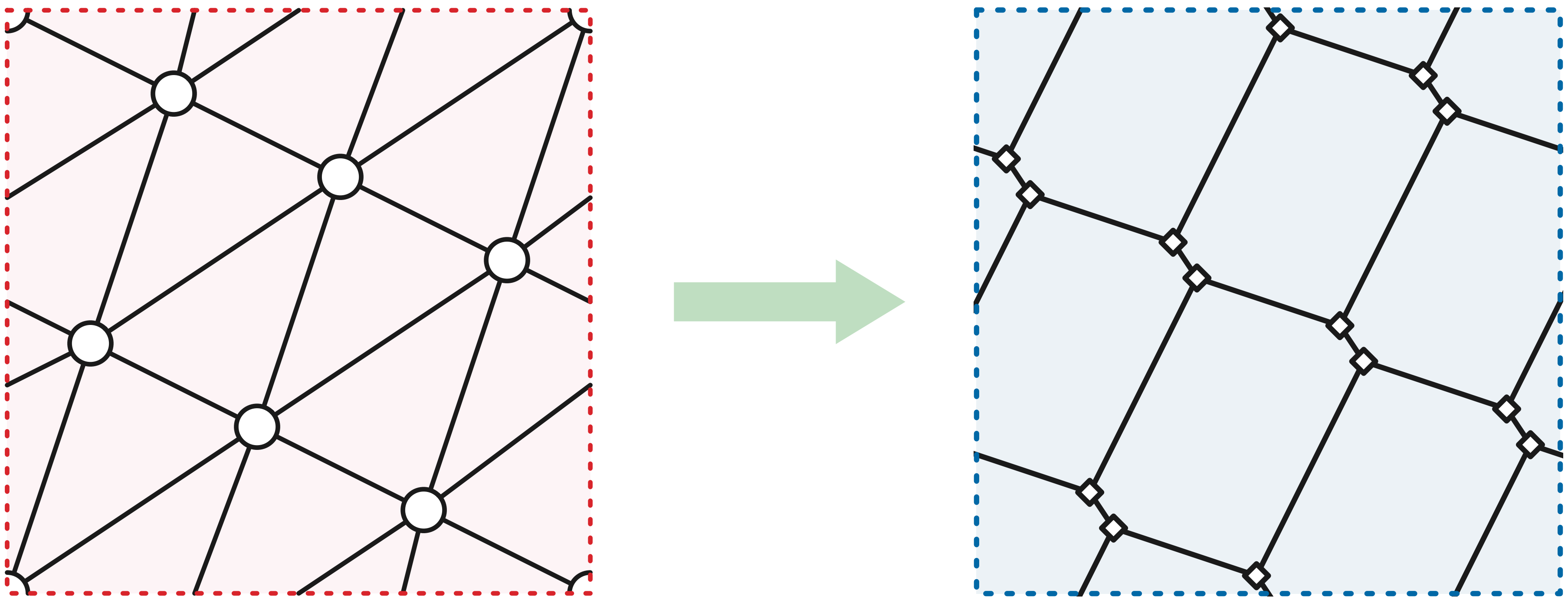


Voronoi

Delaunay ⇔ reciprocal ⇒ equilibrium [E Lin2020]

Every reciprocal diagram defines an equilibrium stress:

$$\omega_e = |e^*| \,/\, |e|$$



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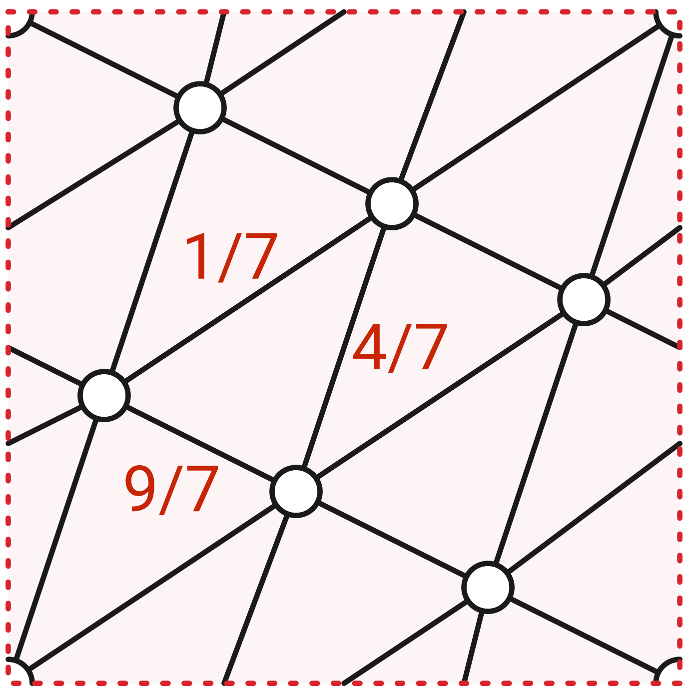
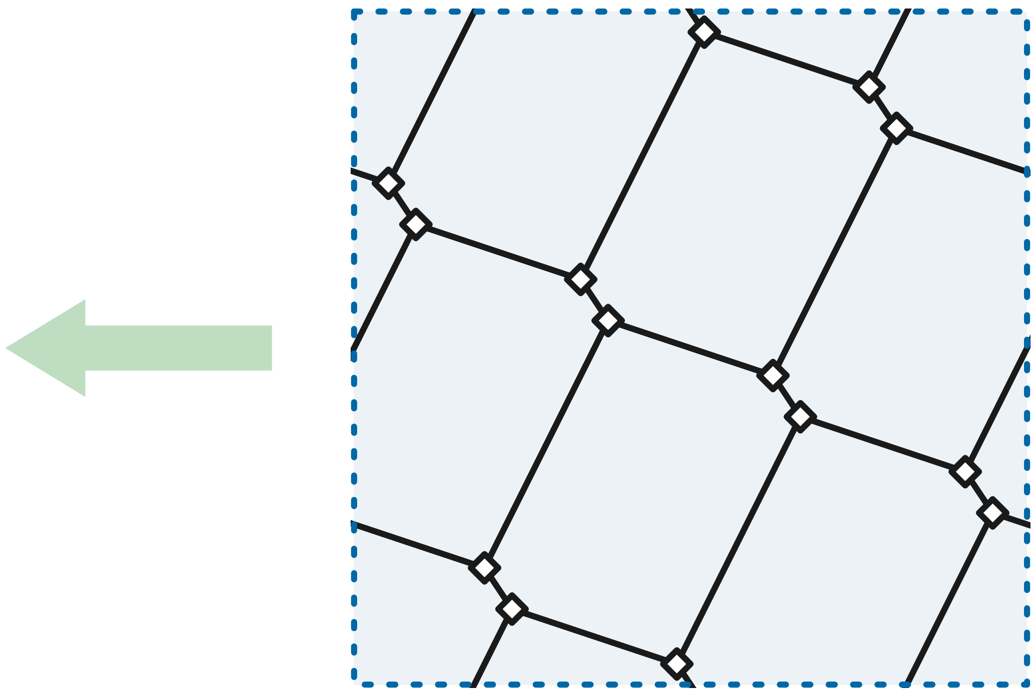
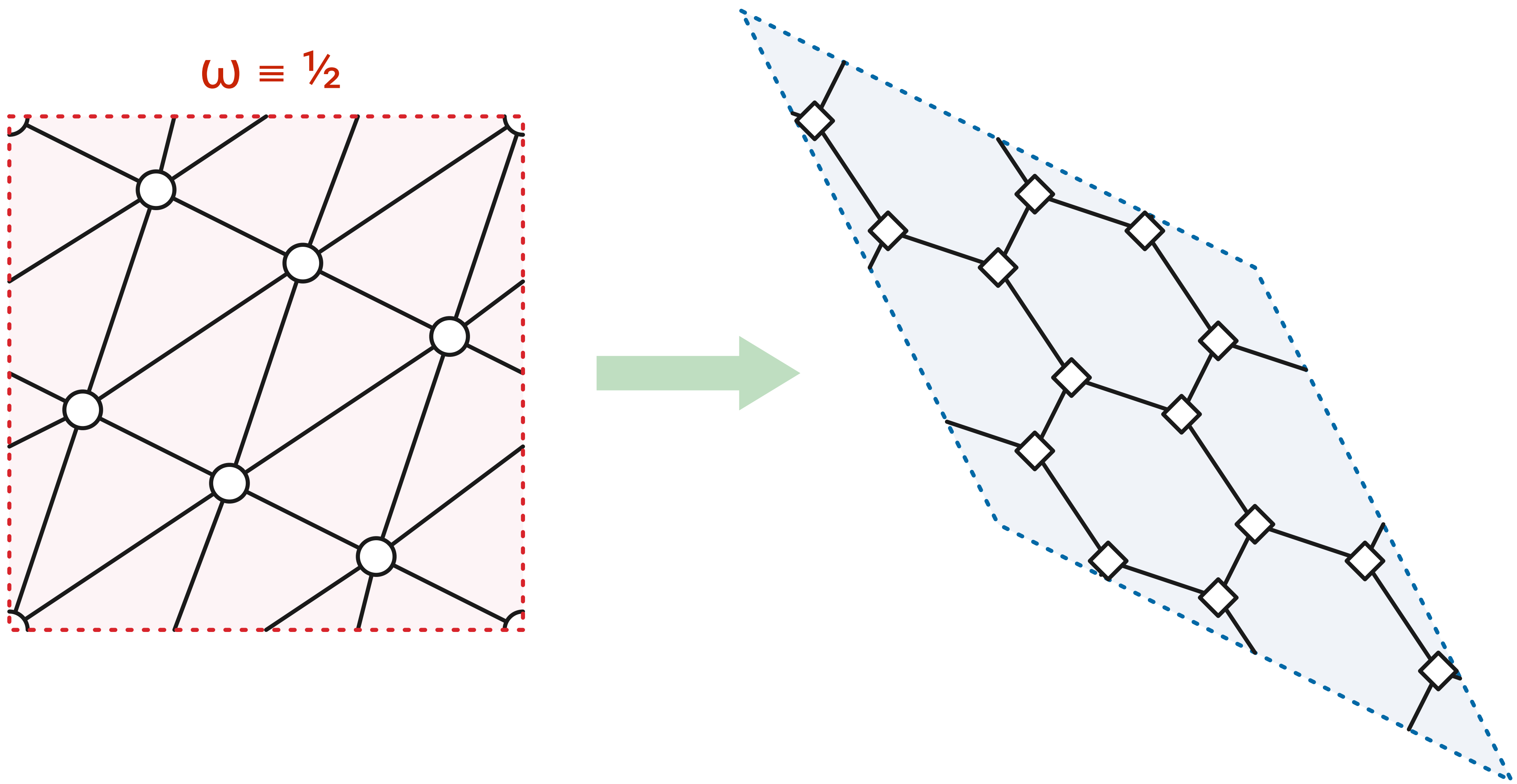


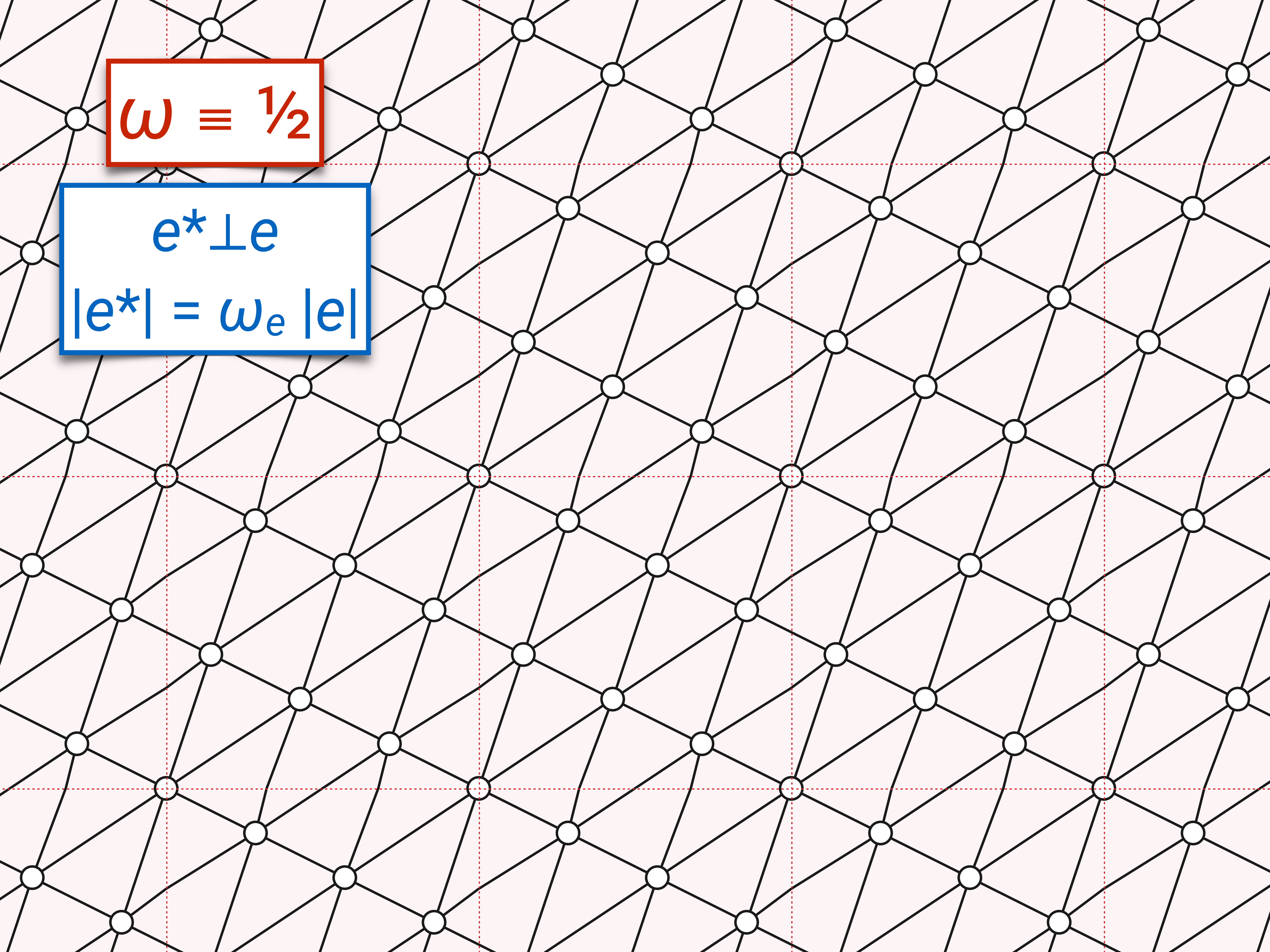


Equilibirum $\not\Rightarrow$ reciprocal

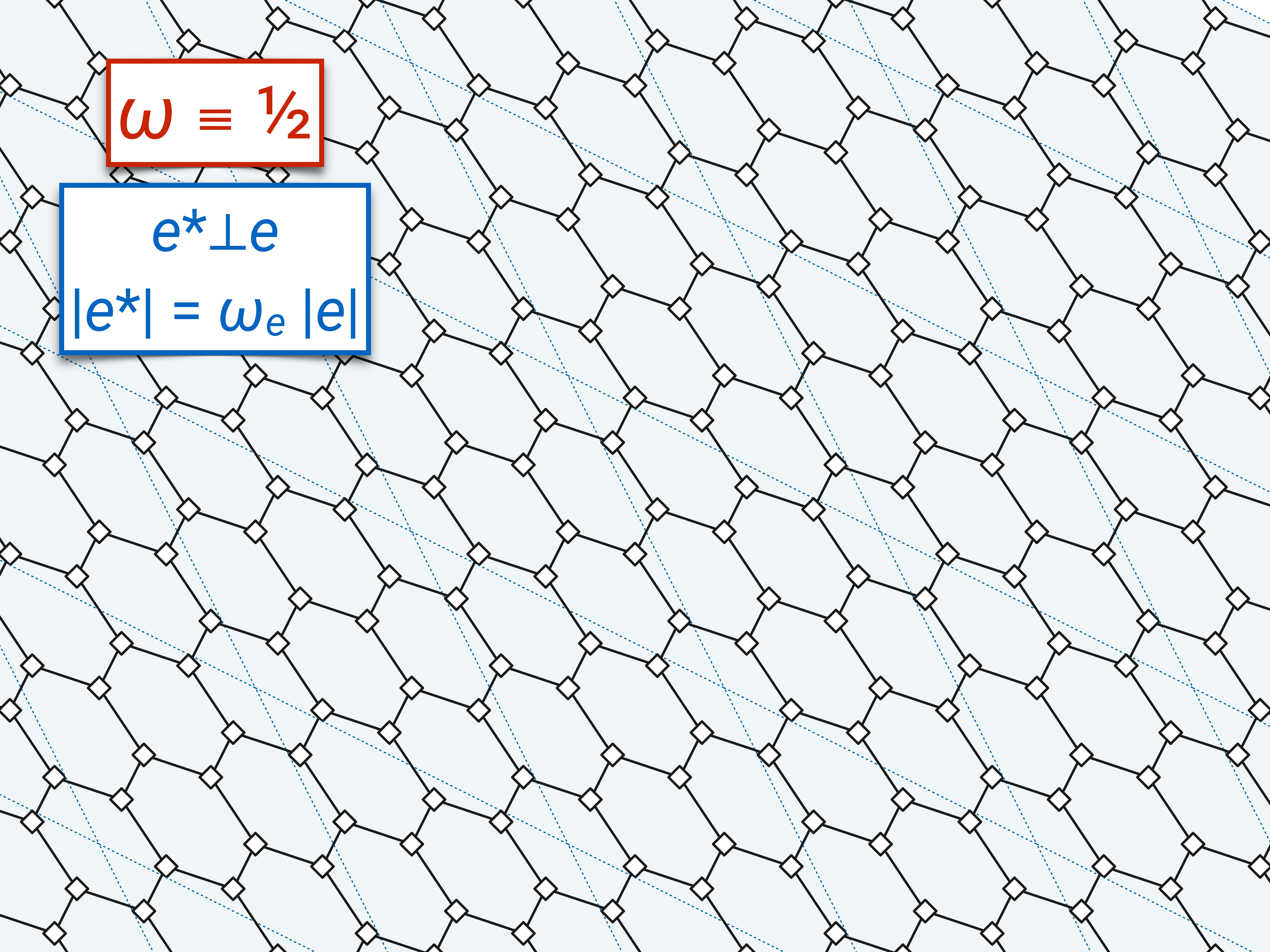
[E Lin 2020]

In general, the *force diagram* defined by an equilibrium stress lies on a *different* flat torus





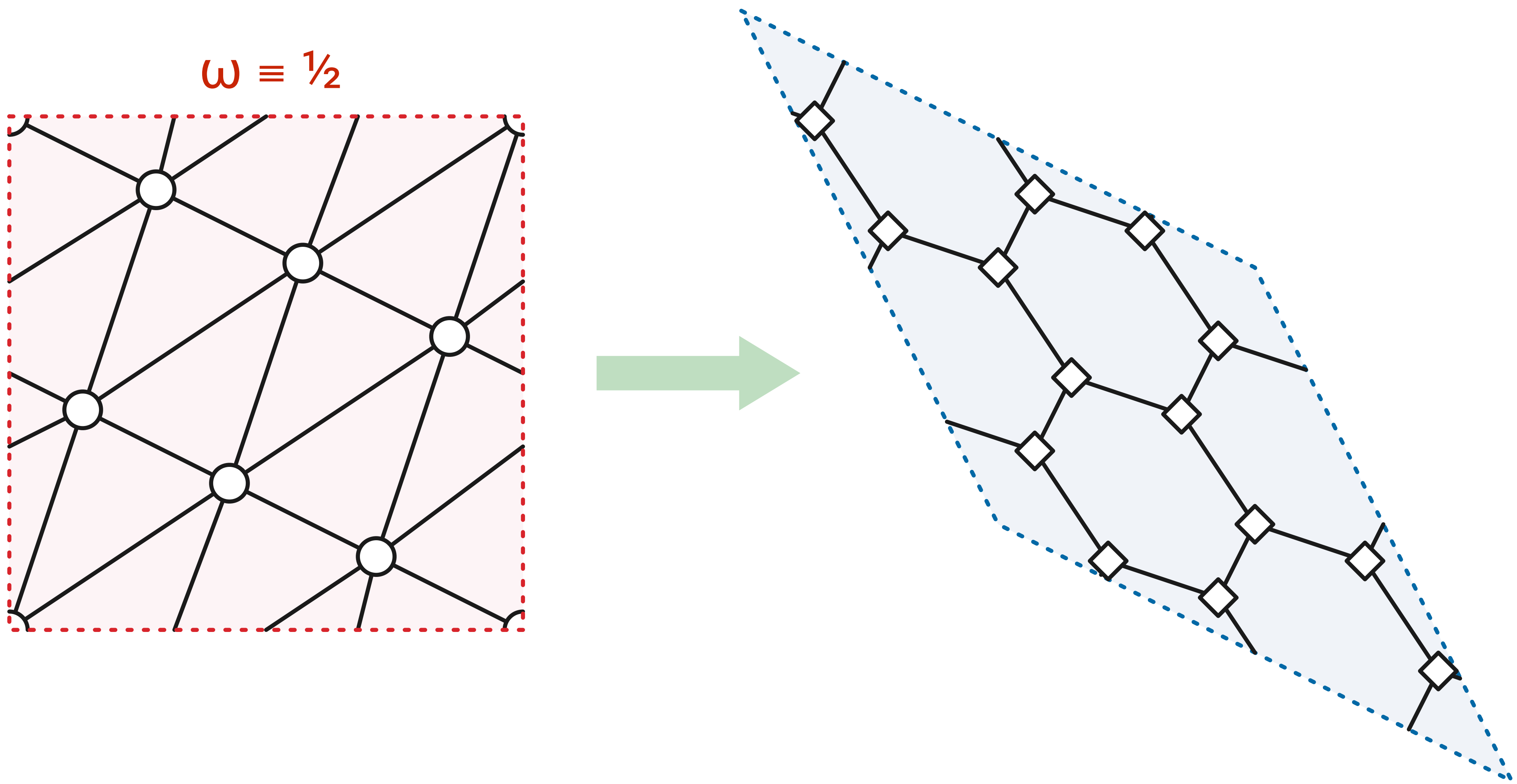
ω ≡ ½
e*⊥e
|e*| = ω_e |e|


$\omega \equiv ½$
$e^* \perp e$
$|e^*| = \omega_e |e|$

Equilibirum $\not\Rightarrow$ reciprocal

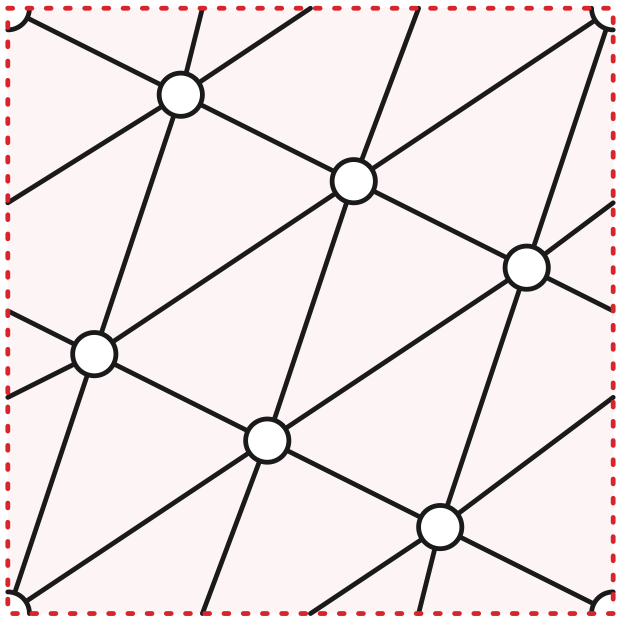
[E Lin 2020]

In general, the *force diagram* defined by an equilibrium stress lies on a *different* flat torus



Equilibirum ⇒ reciprocal somewhere [E Lin 2020]

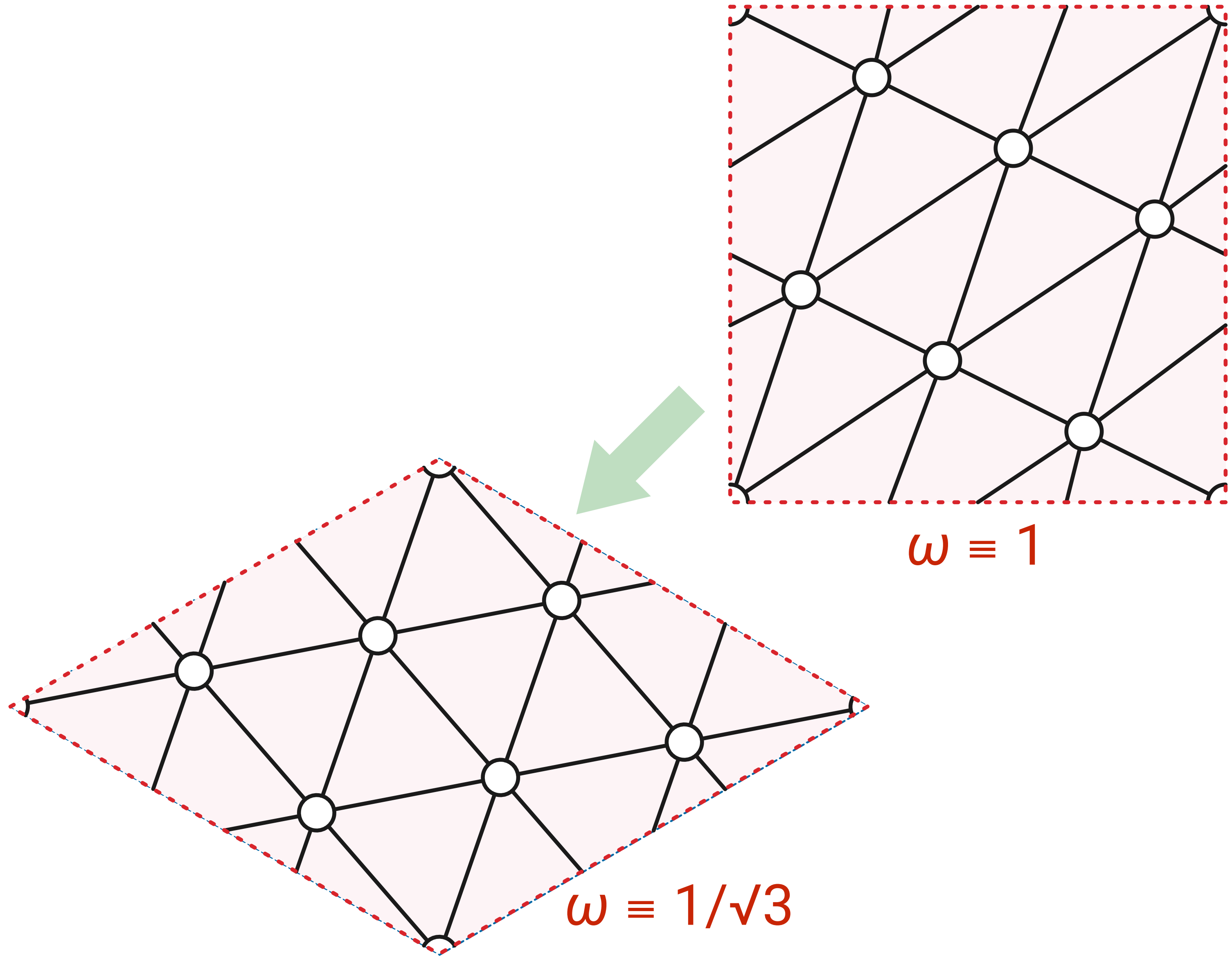
Every equilibirum stress for Γ can be *scaled* to a reciprocal stress for the image of Γ on *some* flat torus.



$\omega \equiv 1$

Equilibirum ⇒ reciprocal somewhere *[E Lin 2020]*

Every equilibirum stress for *Γ* can be ***scaled*** to
a reciprocal stress for the image of *Γ* on ***some*** flat torus.



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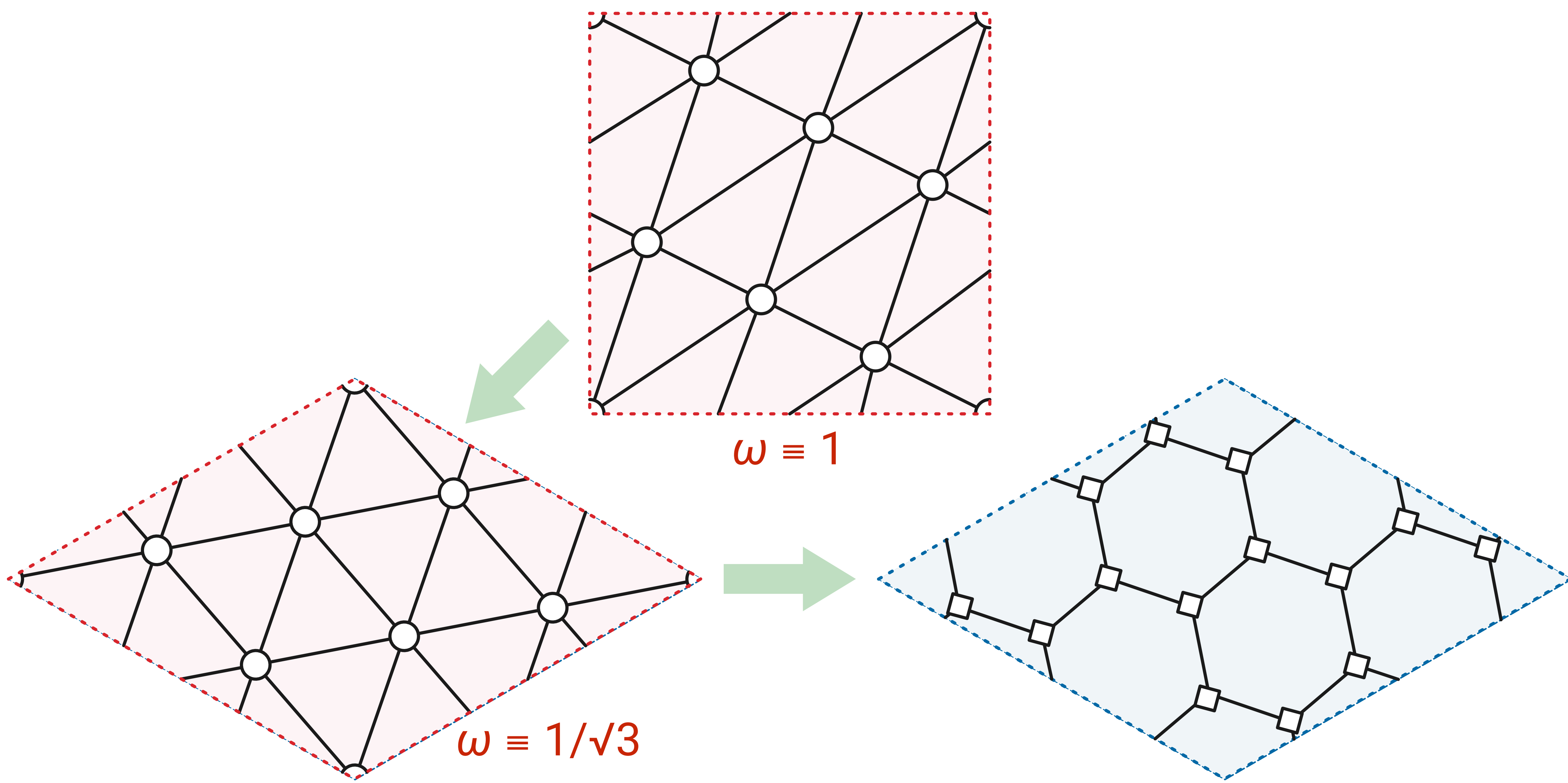


[E Lin 2020]

Maxwell–Cremona–Delaunay correspondence

For any essentially 3-connected geodesic embedding Γ on any flat torus, the following are (essentially) equivalent:

- Positive ***equilibrium stress*** ω for Γ
- Positive ***equilibrium stress*** ω for the image of Γ on *every* flat torus
- *Reciprocal diagram* for the image of Γ on *some* (essentially unique) flat torus
- *Delaunay* vertex weights for the image of Γ on *some* (essentially unique) flat torus

Mighty Morphin Power Rangers donuts (2017)

ANNALS OF MATHEMATICS
Vol. 45, No. 2, April, 1944

ISOTOPIC DEFORMATIONS OF GEODESIC COMPLEXES ON THE 2-SPHERE AND ON THE PLANE[1]

BY STEWART S. CAIRNS

(Received February 9, 1943)

1. The deformation theorem

Two simplicial complexes, K_0 and K_1, are called *isomorphic* if their respective sets of vertices can be so numbered, P_i and Q_i $(i = 1, 2, \cdots)$, that $Q_{i_0} \cdots Q_{i_m}$ is a cell of K_1 when and only when $P_{i_0} \cdots P_{i_m}$ is a cell of K_0. We will then say that the vertices are *similarly numbered.*

A complex on a euclidean 2-sphere will be referred to as *geodesic* if each of its 1-cells is an arc of great circle shorter than a semi-circle and each of its 2-cells is a spherical triangle less in area than a hemisphere. Every complex throughout this paper will be finite and will be non-singular, in the sense that no two different cells of any one complex will have a point in common.

DEFORMATION THEOREM.[2] *Let K_0 and K_1 be a pair of isomorphic geodesic complexes on a euclidean 2-sphere, S. Let P_i and Q_i $(i = 1, \cdots, n)$ be the vertices, similarly numbered, of K_0 and K_1 respectively. If and only if the isomorphism*[3] *between K_0 and K_1 can be extended into an orientation-preserving self-homeomorphism of S, it is possible to define, for every t $(0 \leqq t \leqq 1)$, a geodesic complex, K_t, with vertices $P_i(t)$ $(i = 1, \cdots, n)$ in such a way that* (1) $P_i(0) = P_i$ *and* $P_i(1) =$

Planar morphing theorem

[Cairns 1943]

For any two equivalent planar straight-line embeddings Γ_0 and Γ_1 of the same graph G, with the same convex outer face, there is an geodesic isotopy from Γ_0 to Γ_1.

- *equivalent* = same rotation system = orientably homeomorphic
- *isotopy* = continuous deformation through embeddings
- *geodesic* isotopy = continuous deformation through *straight-line* embeddings = *morph*

[2] Ernst Steinitz proved a similar theorem for convex polyhedra in euclidean 3-space [E. Steinitz and H. Rademacher, *Vorlesungen über die Theorie der Polyeder*, Berlin (1934), p. 347]. If a convex polyhedron has only triangular faces, its projection from an inner point onto a sphere, S, about the point gives a geodesic triangulation of S. One might deduce the present theorem from that of Steinitz by showing (if it be true) that every geodesic triangulation of S is obtainable as a central projection of a convex polyhedron.

[Cairms 44]

über. Damit ist der wichtige *Kontinuitätssatz der konvexen Polyeder* bewiesen (bis auf den noch zu behandelnden Sonderfall der Pyramiden):

Zwei projektiv-konvexe Polyeder von gleichem Typ lassen sich unter Aufrechterhaltung ihrer projektiven Konvexität und ihres Typus stetig ineinander überführen.

[Steinitz Rademacher 34]

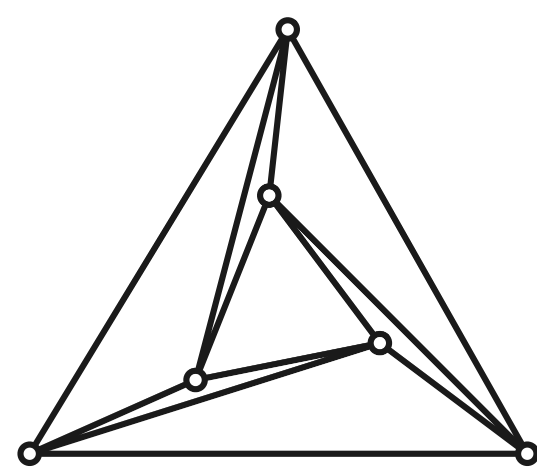
Morphing spring embeddings

[Steinitz Rademacher 1934]

- If Γ_0 and Γ_1 are *equilibrium* embeddings with positive stress vectors $\omega_0 \neq \omega_1$, we can morph by *linearly interpolating stress*:

$$\omega_t = t \cdot \omega_1 + (1-t)\omega_0$$

- Sadly, some plane graphs have *no* positive equilibrium stress.



[Schönhardt 1928]

- We can similarly morph between *equilibrium torus* graphs!

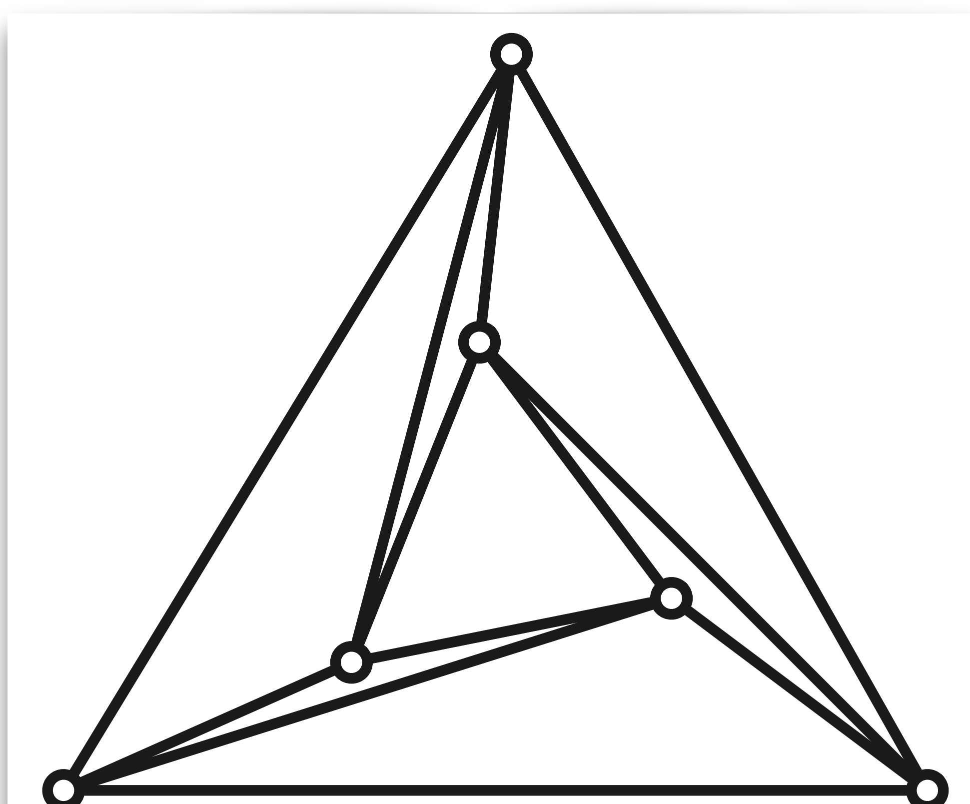
[Éric Colin de Verdière, Pocchiola, Vegter 2003]

Asymmetric springs

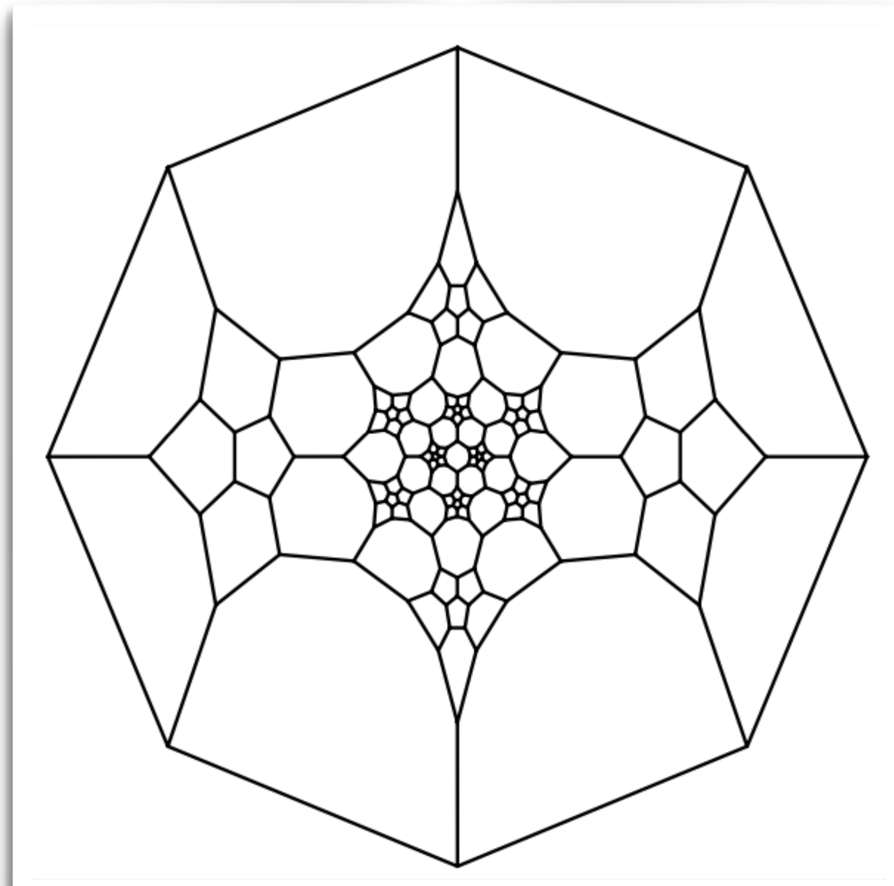
[Floater 1997, 1998, 2000]

In every *convex* planar embedding,
every interior vertex *u* is a weighted average of its neighbors.

$$\sum_v \lambda_{u\to v}(p_v - p_u) = (0,0)$$



[Schönhardt 1928]



[Delgado-Friedrichs 2003]

Asymmetric springs

[Floater 1997, 1998, 2000]

In every *convex* planar embedding,
every interior vertex u is a weighted average of its neighbors:

$$\sum_v \lambda_{u\to v}(p_v - p_u) = (0,0)$$

The ***barycentric weights*** $\boldsymbol{\lambda_{u\to v}}$ could be asymmetric: $\lambda_{u\to v} \neq \lambda_{v\to u}$

But Tutte's proof doesn't care.

Every positive barycentric weight vector
yields a convex planar embedding!

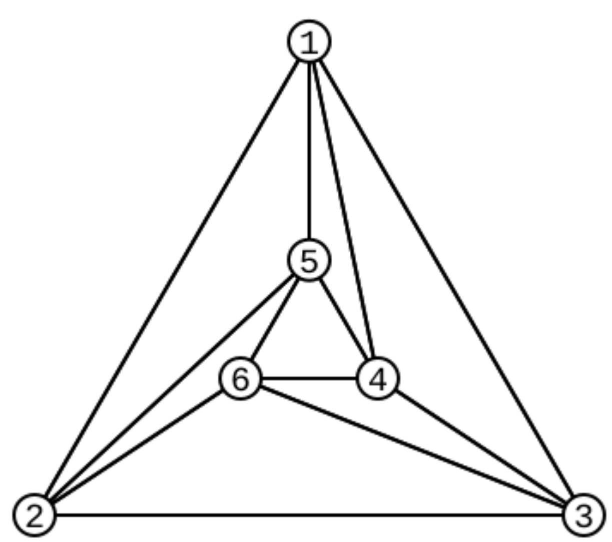
Barycentric interpolation

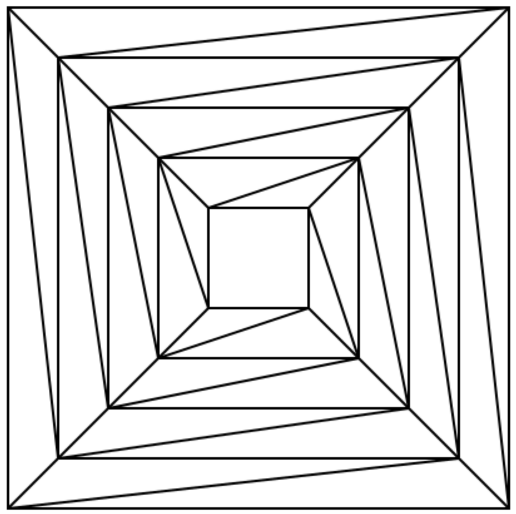
[Floater Gotsman 1999]

Positive barycentric weights ⇔ convex planar embeddings

$$\sum_{v} \lambda_{u\to v}(p_v - p_u) = (0,0)$$

Linearly interpolating between barycentric coordinates yields a morph through convex embeddings





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[Floater Gotsman 1999]

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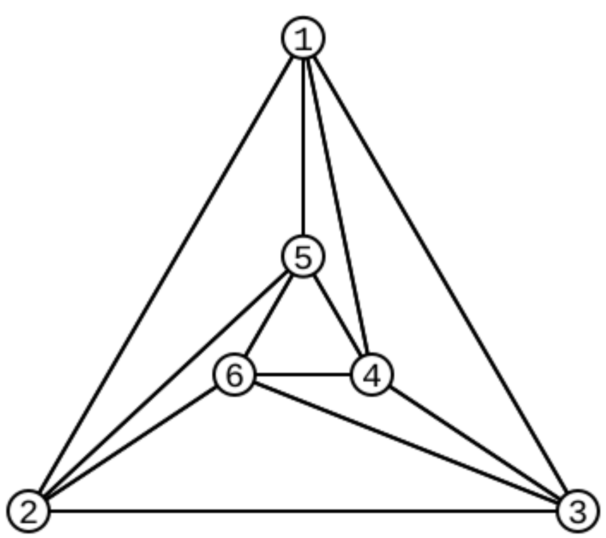

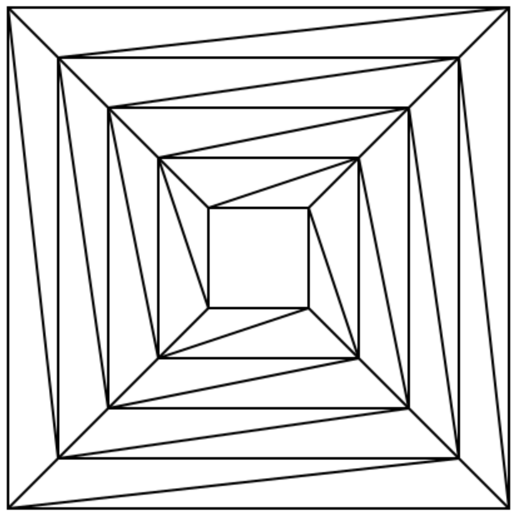

Does this work on the torus?

$$\sum_{v} \lambda_{u\to v}(p_v - p_u + \tau_{u\to v}) = (0,0)$$

- The equilibrium linear system has rank $2n-2$.
- ***IF*** the system is solvable, then it has a two-dimensional family of solutions = all translations of the same convex embedding.

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- The equilibrium linear system has rank $2n-2$.
- ***IF*** the system is solvable, then it has a two-dimensional family of solutions = all translations of the same convex embedding.
- The system is solvable ***IF*** weights are symmetric: $\lambda_{u\to v} = \lambda_{v\to u}$

No, it doesn't.

[Steiner Fischer 2004]

$$\sum_v \lambda_{u\to v}(p_v - p_u + \tau_{u\to v}) = (0,0)$$

- The equilibrium linear system has rank $2n-2$.
- Unfortunately, this system is ***not solvable*** in general. ☹️

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- The equilibrium linear system has rank $2n-2$.
- Unfortunately, this system is ***not solvable*** in general. ☹️
- Worse, averages of realizable barycentric coordinates are not necessarily realizable! 😫

Sometimes it does.

[E Lin 2020]

$$\sum_{v} \lambda_{u\to v}(p_v - p_u + \tau_{u\to v}) = (0,0)$$

- Rewrite the equilibrium system in matrix form: $L_\lambda P = T_\lambda$
- Call the weight vector λ ***morphable*** if every column of L_λ sums to zero and every column of T_λ sums to zero.

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- Call the weight vector λ *morphable* if every column of L_λ sums to zero and every column of T_λ sums to zero.

- Easy Lemma 1: Every morphable weight vector is realizable.
- Easy Lemma 2: Averages of morphable weight vectors are morphable.

Linear algebra FTW!

[E Lin 2020]

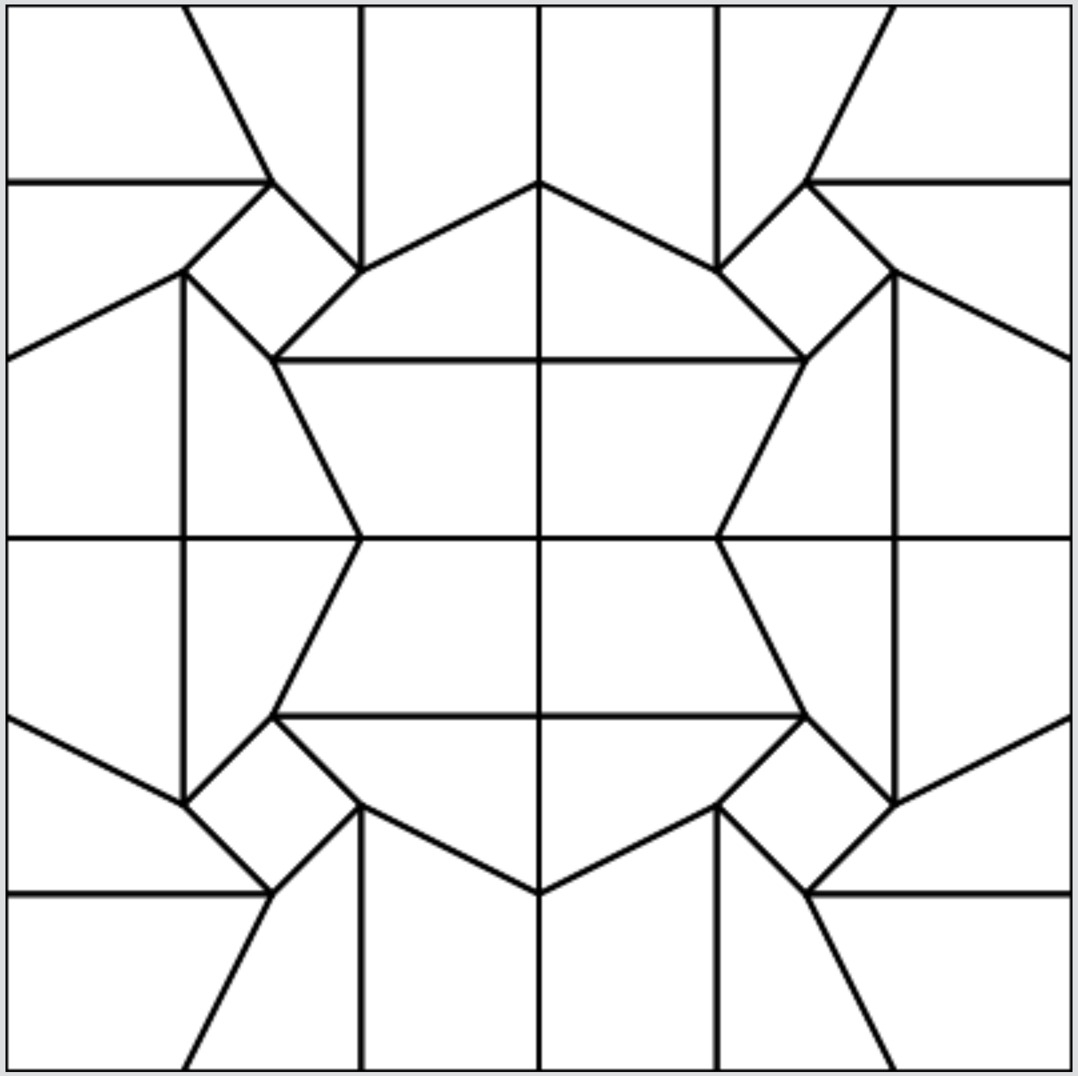
- Equilibrium linear system: $L_\lambda P = T_\lambda$
- λ is ***morphable*** if every column of L_λ sums to zero and every column of T_λ sums to zero.

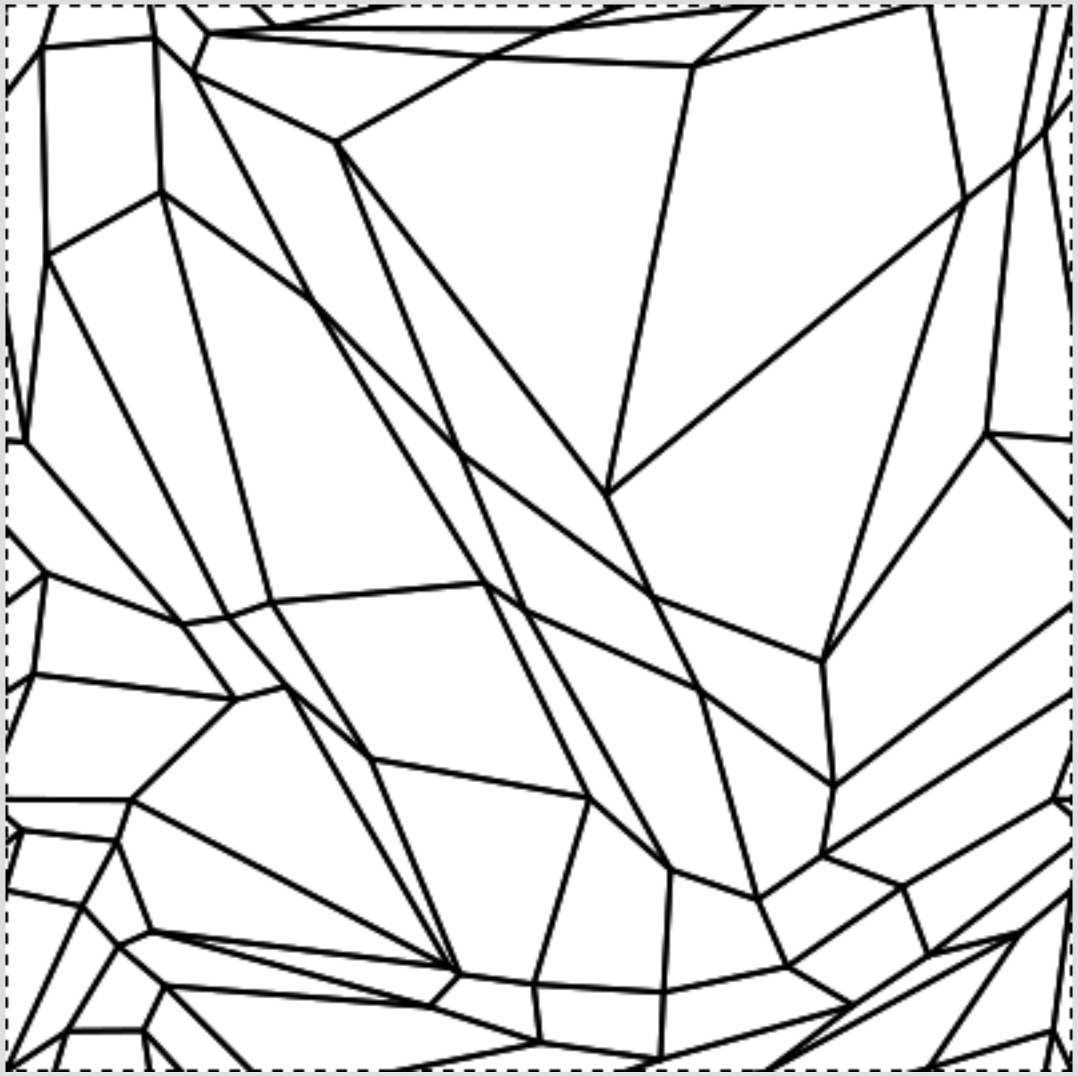
- **Slightly Harder Lemma:** Any barycentric weight vector for a convex embedding Γ can be ***scaled*** to a morphable weight vector for the same embedding Γ.
  - Let $\mu_{u\to v} = \alpha_u \lambda_{u\to v}$, where α is any left null vector of L_λ.
  - Matrix-Tree Theorem (or Perron-Frobenius) implies wlog $\alpha_u > 0$ for all u.

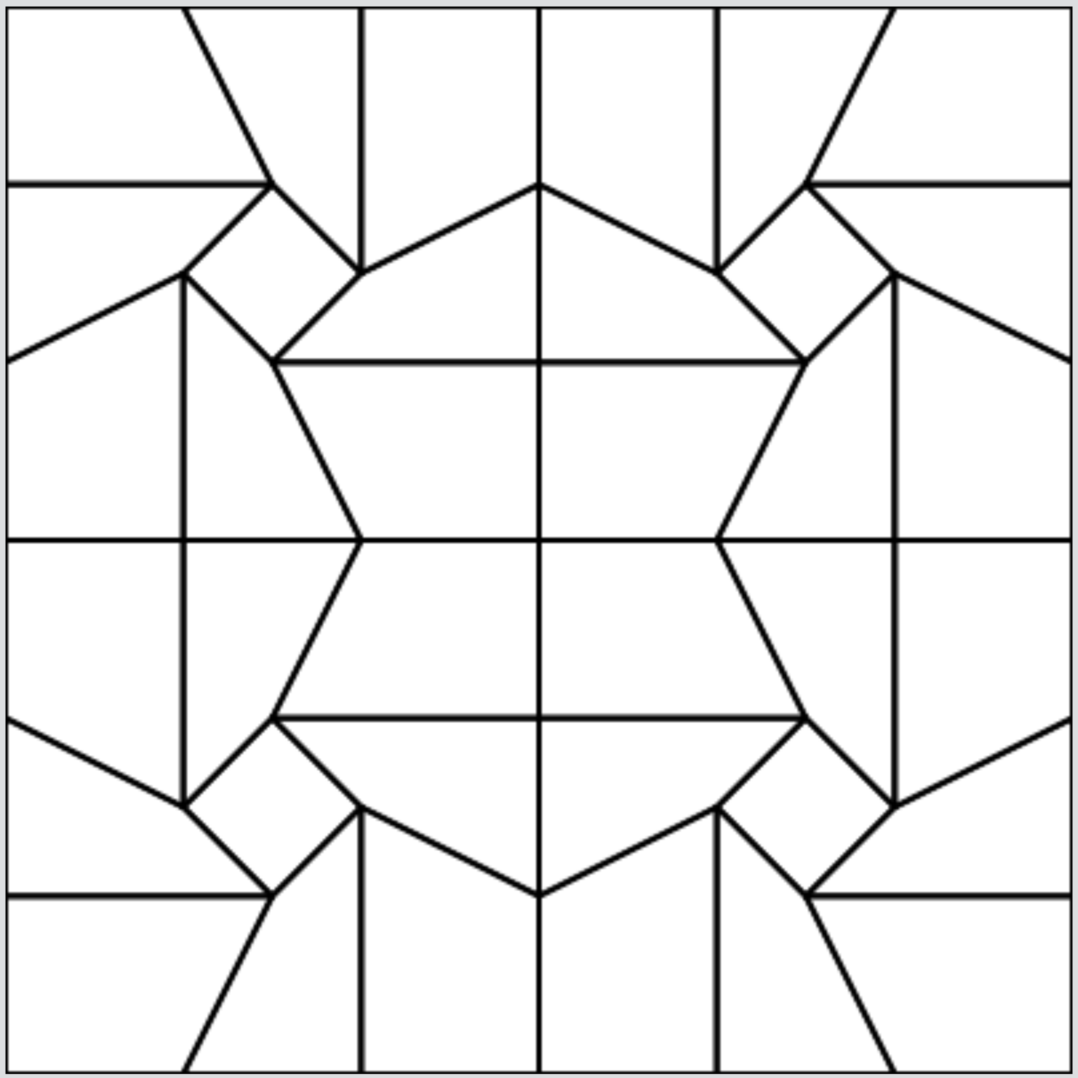
Mighty morphin torus graphs

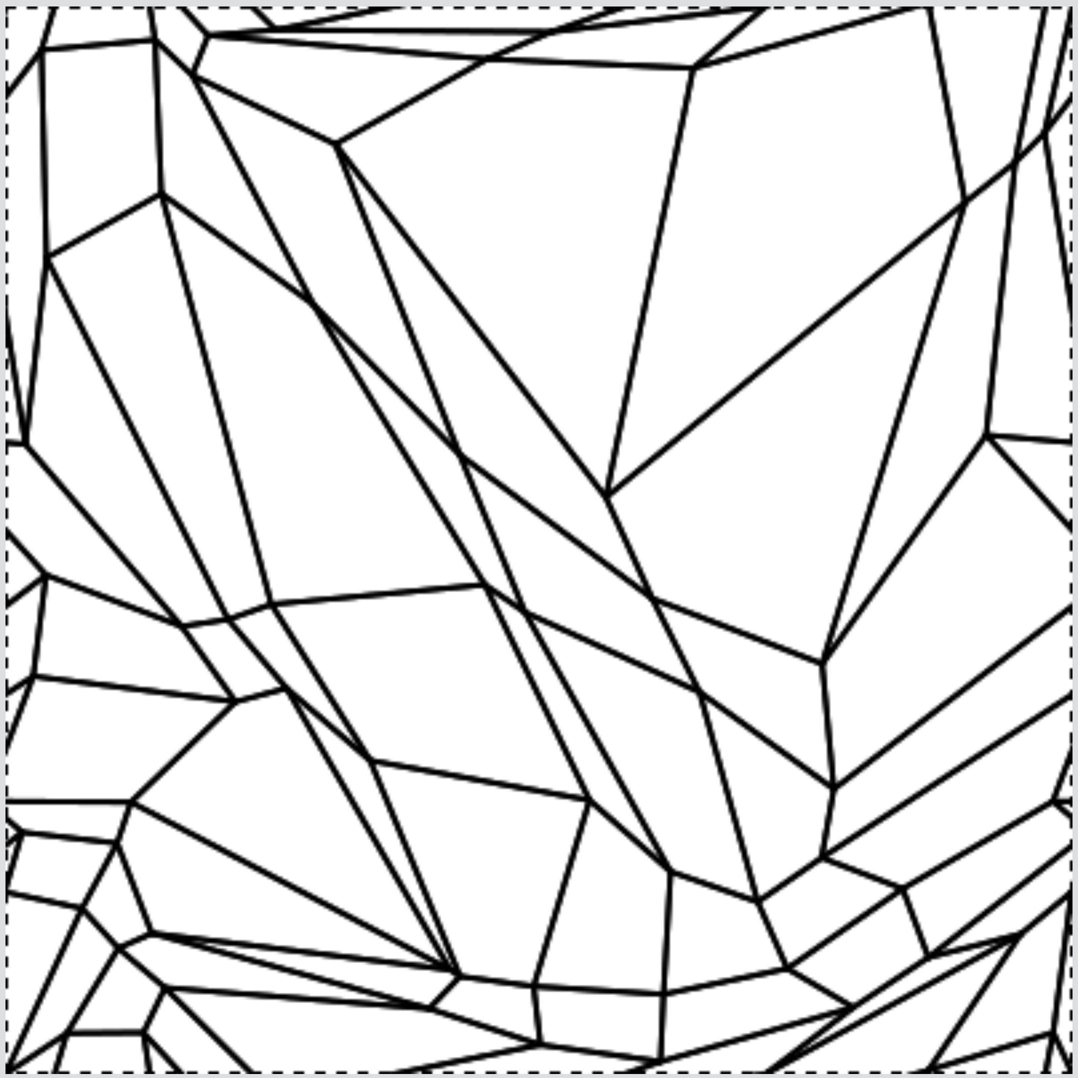
[E Lin 2020]

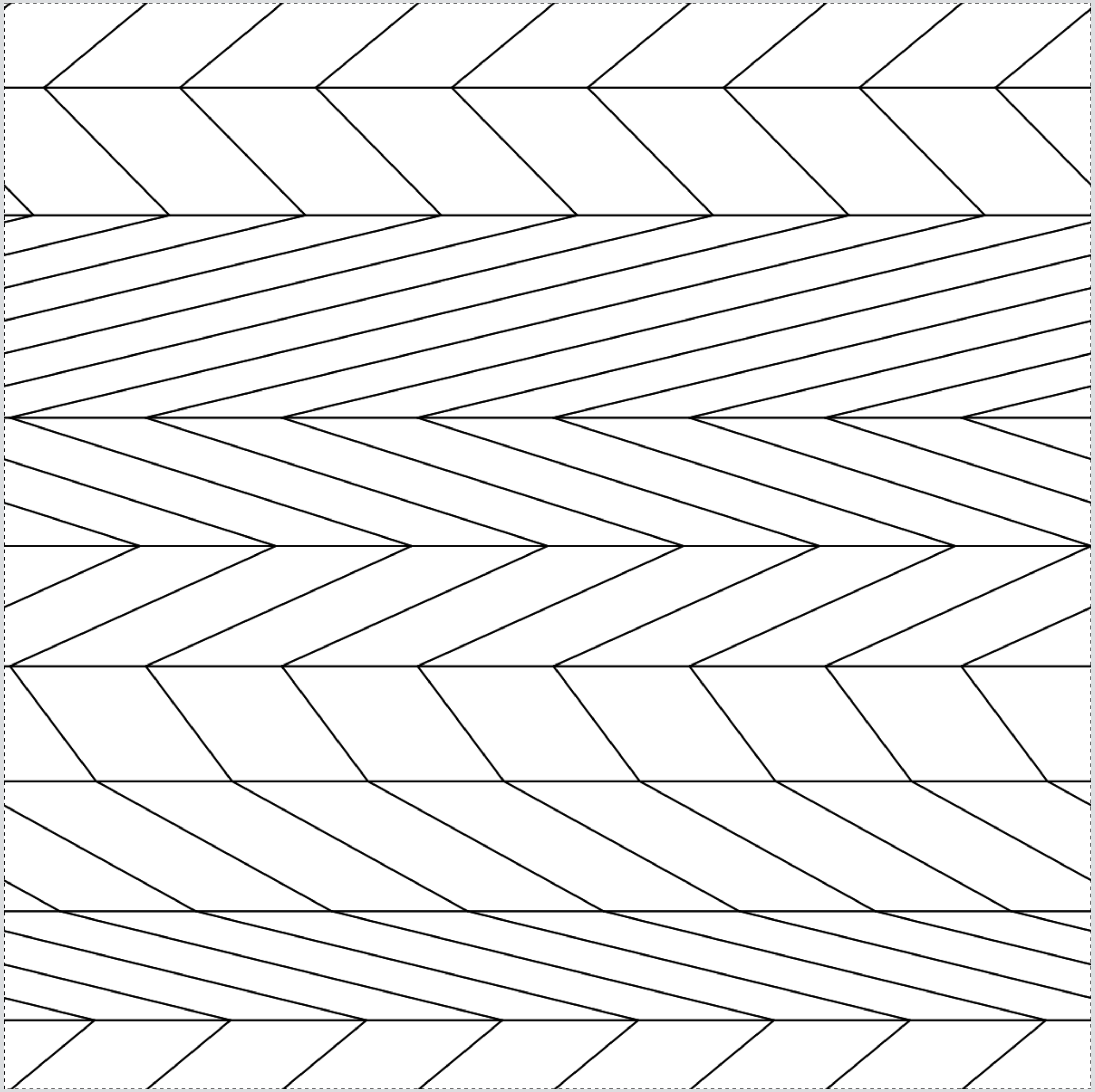
- Given two geodesic embeddings Γ_0 and Γ_1 on the flat torus.
- Check whether Γ_0 and Γ_1 are isotopic in $O(n)$ time.
  [Colin de Verdière, de Mesmay 2014][Chambers E Lin Parsa 2020]
- Compute barycentric weight vectors λ_0 and λ_1 in $O(n)$ time.
  [Floater 2000]
- Scale to morphable weight vectors μ_0 and μ_1 in $O(n^{\omega/2})$ time.
- For any $0<t<1$:
  - compute intermediate morphable weights $\mu_t = \mu_0(1-t) + \mu_1 \cdot t$
  - compute embedding Γ_t by solving equilibrium system in $O(n^{\omega/2})$ time.

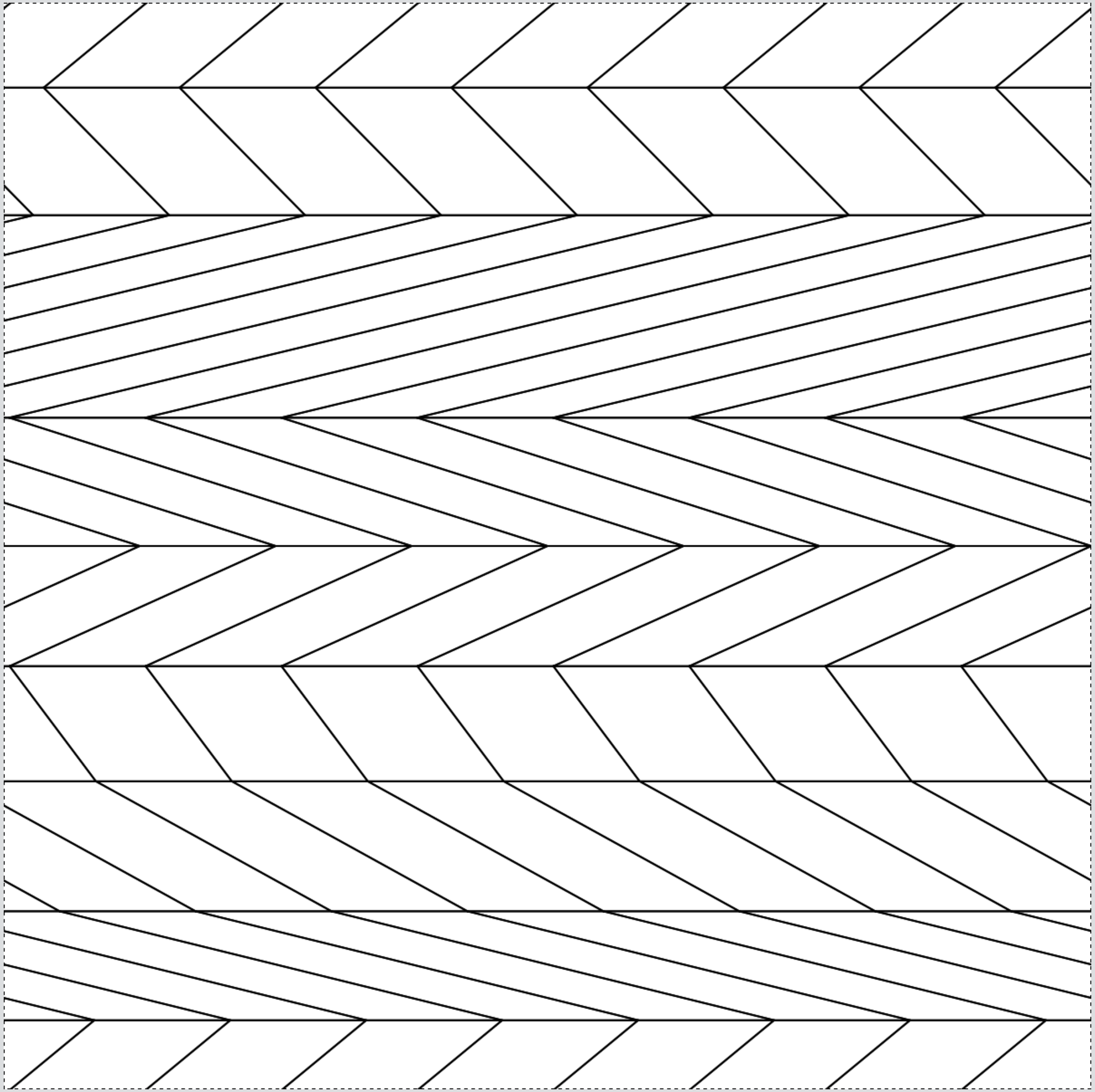












Extensions and open problems

- We can morph between isotopic geodesic embeddings on any surface with *negative curvature* via barycentric interpolation (*without* scaling!) *[Luo Wu Zhu 2021]*

- Can we morph geodesic embeddings on the *sphere*?
  - C*oherent* embeddings: Yes, via Tutte and Maxwell-Cremona! *[Richter-Gebert 1996]*
  - *Convex* embeddings: Yes(?), via edge contractions. *[Cairns 1944]*
  - *Arbitrary* embeddings? *Open!*

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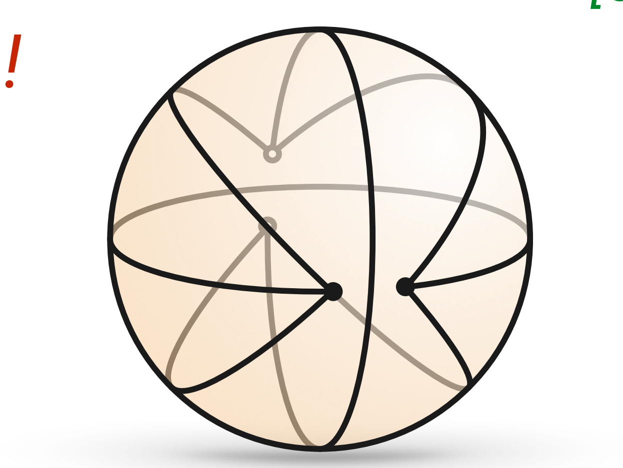
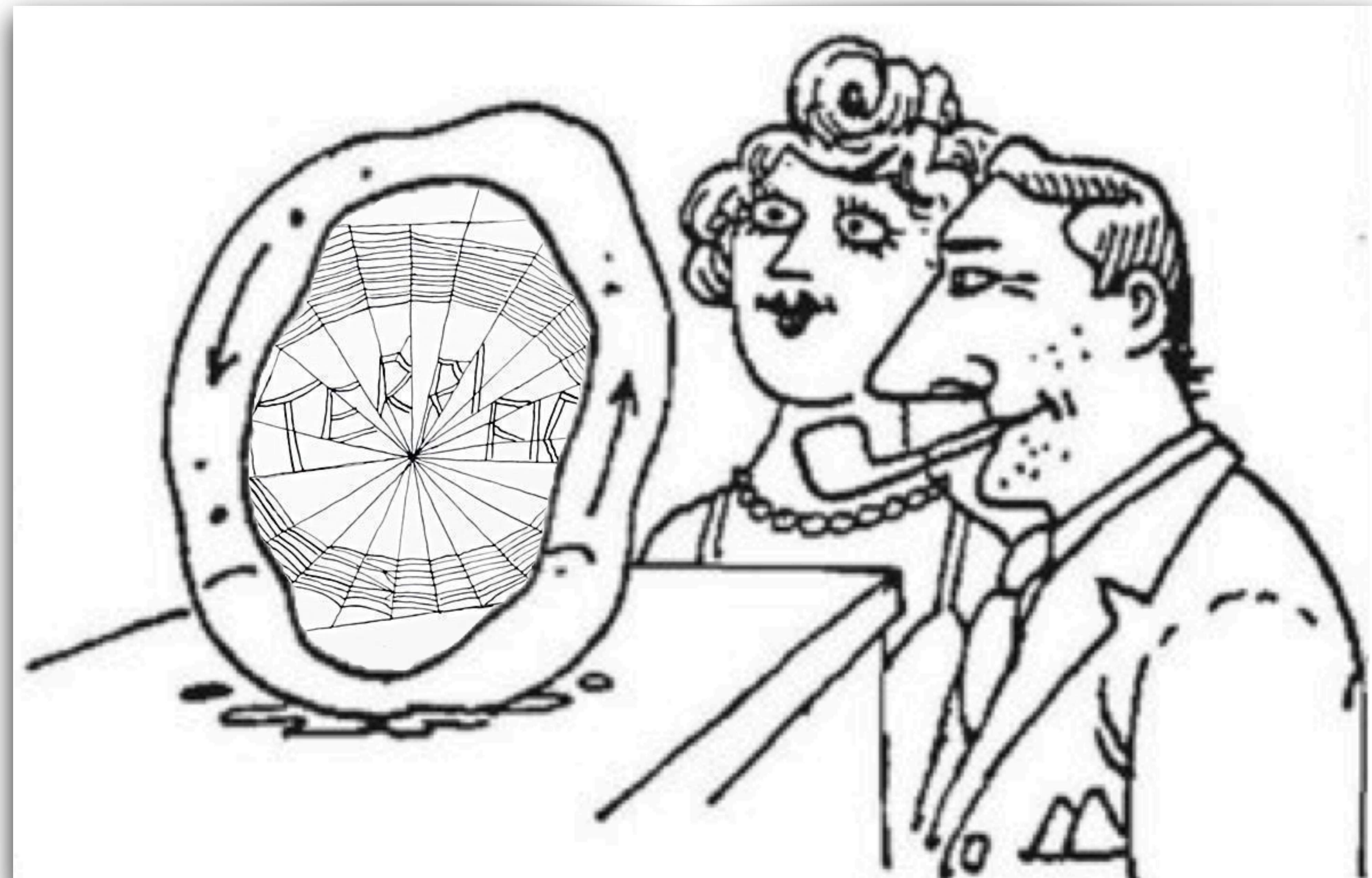


Thank you!



Thank you!