Enumeration of square-tiled surfaces and metric ribbon graphs

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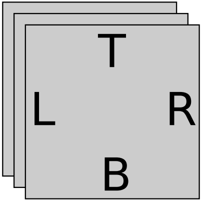
May 2, 2022

Square-tiled surfaces



T $\leftrightarrow$ B, L $\leftrightarrow$ R

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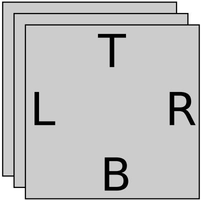

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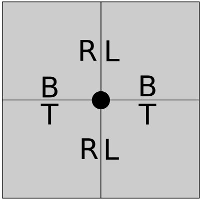
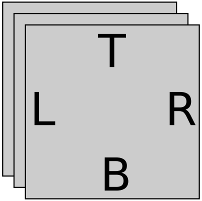


2π – flat point

Square-tiled surfaces



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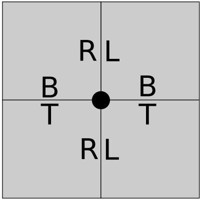

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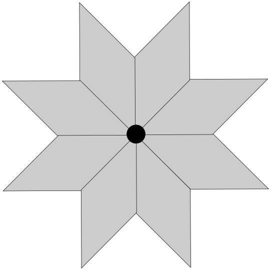
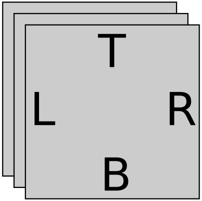


4π – singularity

Square-tiled surfaces



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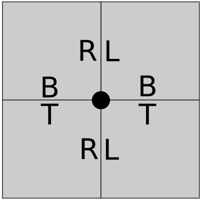

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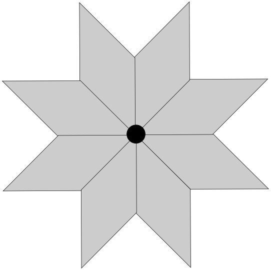

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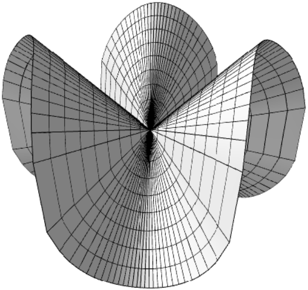


Counting square-tiled surfaces

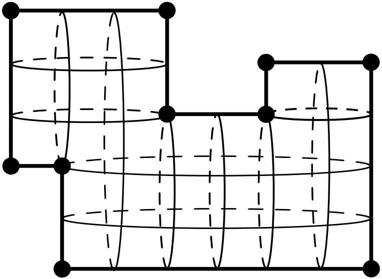
I am interested in counting square-tiled surfaces with fixed number and angles of singularities $2\pi(k_1+1), \ldots, 2\pi(k_n+1)$.
More precisely, the limit

$$\lim_{N \to +\infty} \frac{|\mathcal{ST}(k,N)|}{N^{2g+n-1}},$$

where

- $k = (k_1, \ldots, k_n)$
- g is the corresponding genus, $k_1 + \ldots + k_n = 2g-2$;
- $\mathcal{ST}(k,N)$ is the set of surfaces with such singularities and at most N squares.

Cylinder decomposition



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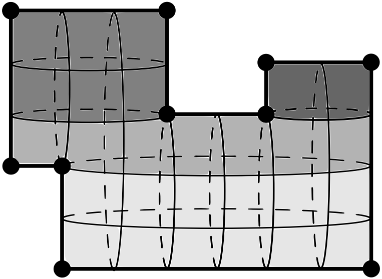

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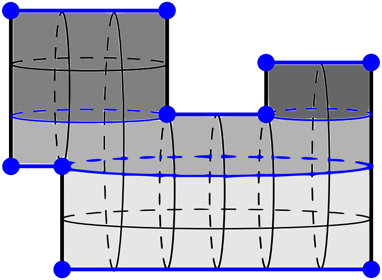

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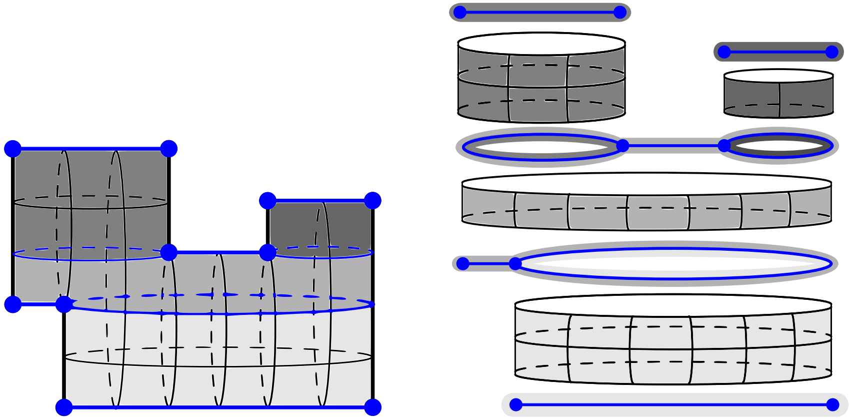
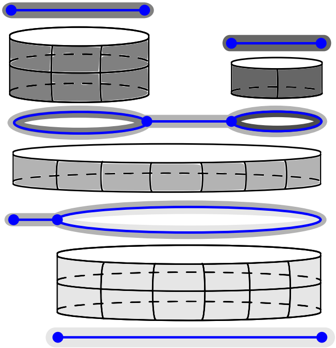


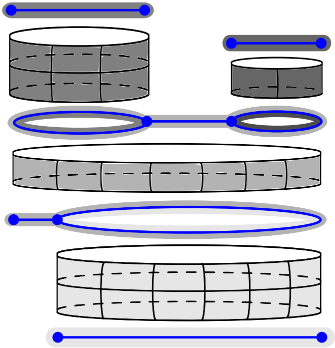
Square-tiled surface = cylinders + ribbon graphs.

Back to counting



- Cylinders are easy to count: height $h_i \in \mathbb{Z}_{>0}$, circumference $L_i \in \mathbb{Z}_{>0}$.

Back to counting



- Cylinders are easy to count: height $h_i \in \mathbb{Z}_{>0}$, circumference $L_i \in \mathbb{Z}_{>0}$.
- Remains to count *metric* ribbon graphs of genus g with n boundary components of given perimeters $L_1, \ldots, L_n \in \mathbb{Z}_{>0}$.

Counting metric ribbon graphs

- For a fixed ribbon graph G the number of metrics which give the boundary components the perimeters $L_1, \ldots, L_n$ is a **piecewise quasi-polynomial** $\mathcal{N}_G(L_1, \ldots, L_n)$.

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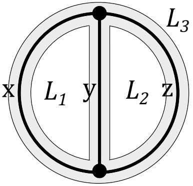
- Example:



$$\begin{cases} x + y = L_1 \\ y + z = L_2 \\ z + x = L_3 \\ x, y, z > 0 \end{cases} \Rightarrow \begin{cases} x = (L_1 + L_3 - L_2)/2 \\ y = (L_2 + L_1 - L_3)/2 \\ z = (L_3 + L_2 - L_1)/2 \\ x, y, z > 0 \end{cases}$$

If $L_1 + L_2 + L_3$ is odd, then $\mathcal{N}_G = 0$.
If $L_1 + L_2 + L_3$ is even, then $\mathcal{N}_G = 1$ if L_1, L_2, L_3 satisfy the triangle inequalities, and $\mathcal{N}_G = 0$ otherwise.

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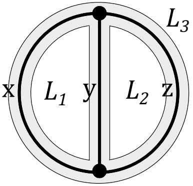

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- However, sometimes miracles happen...

Counting metric ribbon graphs

Theorem (Kontsevich)

Let $L_1 + \cdots + L_n$ be even. The weighted count of **trivalent** metric ribbon graphs of genus g with n boundaries of perimeters $L_1, \ldots, L_n$ is

$$\mathcal{N}_{g,n}(L_1, \ldots, L_n) = N_{g,n}(L_1, \ldots, L_n) + \text{lower order terms},$$

where $N_{g,n}$ is a homogeneous **polynomial**, whose **coefficients are intersection numbers** of psi-classes on the moduli space of curves $\mathcal{M}_{g,n}$.

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Theorem (Y.)

The count of **one-vertex, face-bipartite** metric ribbon graphs of genus g with n black and n white boundaries of equal perimeters $L_1, \ldots, L_n$ is

$$\mathcal{Q}_{g,n}(L_1, \ldots, L_n) = Q_{g,n}(L_1, \ldots, L_n) + \text{lower order terms},$$

where $Q_{g,n}$ is a homogeneous **polynomial**, whose **coefficients enumerate certain families of metric plane trees**.