Joint with Hrant Hakobyan and Dragomir Šarić

Structures on surfaces

Complex Analysis		Hyperbolic Geometry
- Riemann surface	$\longleftrightarrow$	(geodesically) complete hyperbolic surface
- extremal length	$\longleftrightarrow$	geometric length
- parabolic type	$\longleftrightarrow$	ergodicity of geodesic flow

X - Riemann surface

<u>defn:</u>

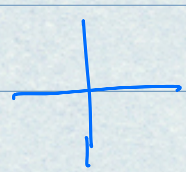
X is <u>parabolic type</u> if it does not support a non-constant negative subharmonic function.

<u>Remark:</u> Not to be confused with parabolic geometry.

- If X is a planar domain then X is parabolic

  $\iff$ any bounded harmonic function is constant

<u>ex</u> unit disc $\subset \mathbb{C}$ <u>not</u> <u>parabolic type</u>

<u>ex</u> $\mathbb{C}$ =  <u>parabolic</u> <u>type</u>

classical

<u>Type Problem:</u> Give necessary and sufficient conditions for X to be of parabolic type.

If $X = \mathbb{H}/G$, $G \underset{\text{torsion-free}}{\overset{\text{discrete}}{\leq}} \mathrm{Isom}^+(\mathbb{H})$

Then TFAE

(1) X is parabolic type

(2) harmonic measure of ideal boundary vanishes.

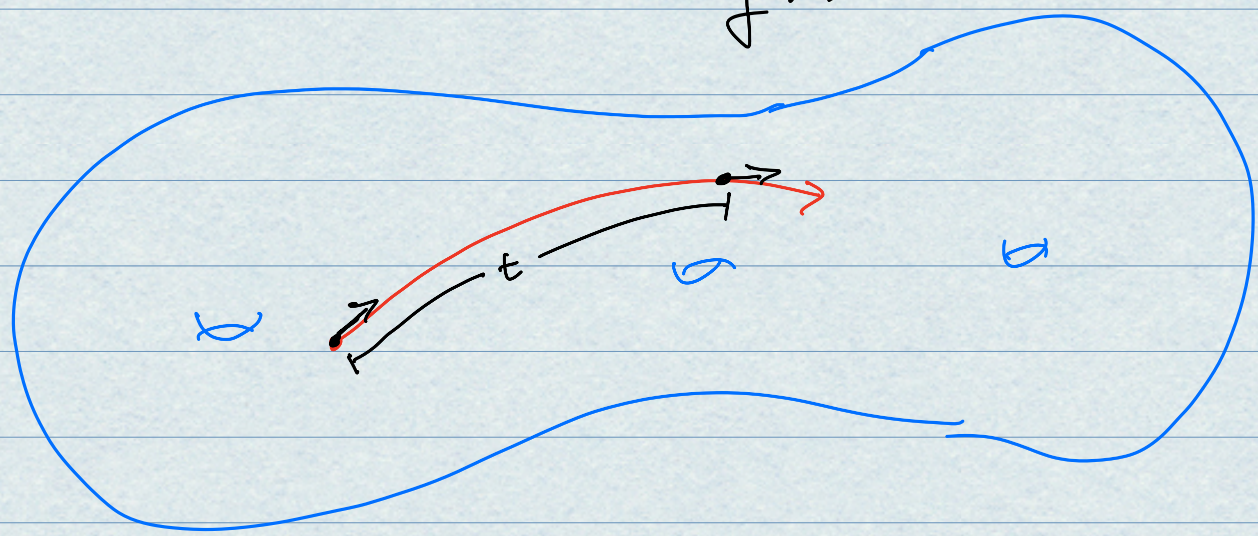
(3) Poincare series of G diverges.

(4) Geodesic flow on the unit tangent bundle is ergodic.

Geometry: X (geodesically) complete hyperbolic surface.

- geodesic flow on the unit tangent bundle

Locally $\mathbb{H}^2 \times S^1$

$X =$ 

Liouville measure on X: $dA\, d\theta$

dA: area measure

$d\theta$: angle measure

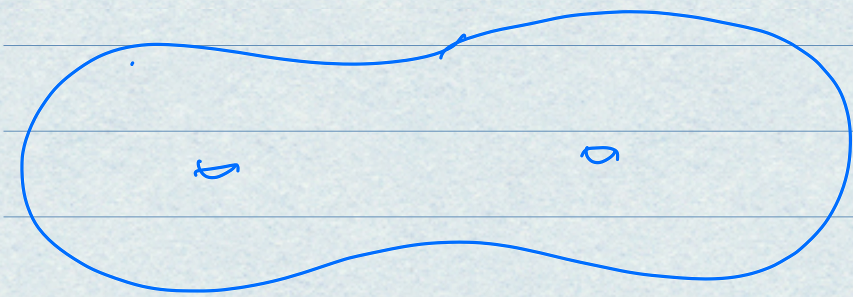
- Geodesic flow on unit tangent bundle is measure preserving.

Def'n: Geodesic flow is said to act ergodically on the unit tangent bundle if invariant sets have measure zero or have complementary set measure zero.

Visual: Dense orbit

Finite topological type ($\pi_1(X)$ f.g.) (well-known)

1) X finite area $\Rightarrow$ parabolic type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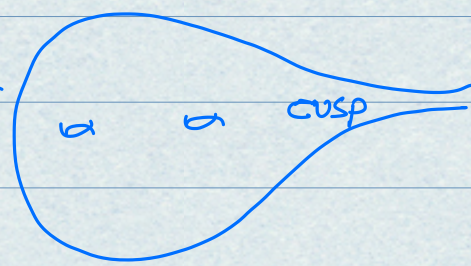


2) X infinite area $\Rightarrow$ not parabolic type

Type Problem (geometric version)

Find necessary & sufficient conditions for geodesic flow to be ergodic.

Our goal:

- Find sufficient conditions (in terms of F-N parameters) that X is of parabolic type.

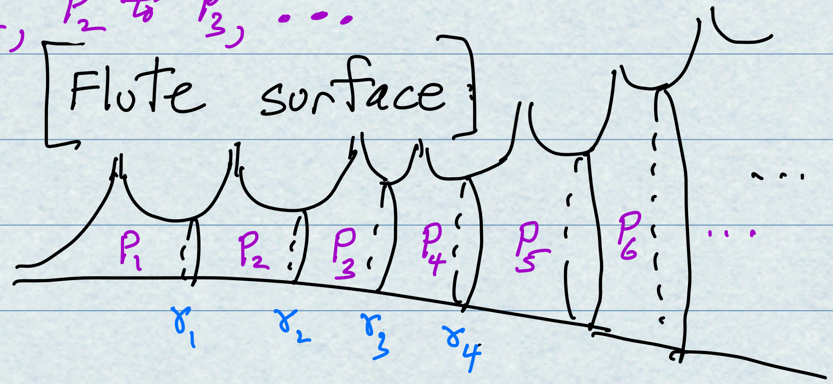
Our focus today is on flute surfaces

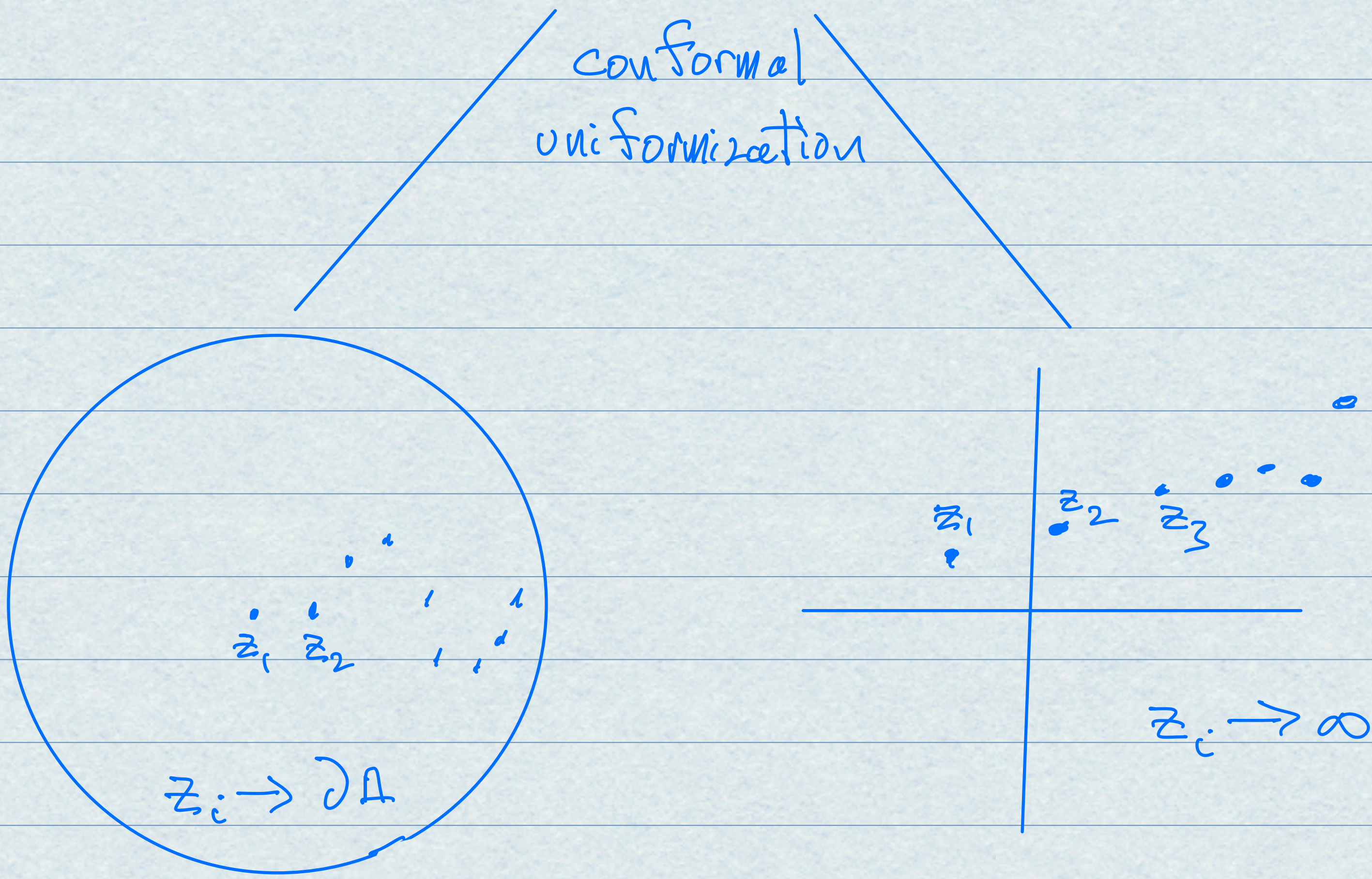
Constructing a flute surface

Glue P_1 to P_2, P_2 to P_3, ...

[Flute surface]

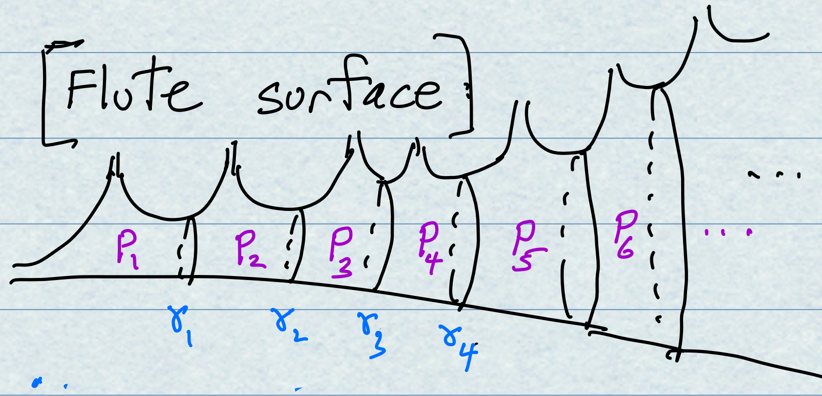
$X(\{\ell_n, t_n\}) =$





$\Delta - \{z_i\}$
(not Parabolic type)

$\mathbb{C} - \{z_i\}$
(Parabolic type)

$X(\{\ell_n, t_n\}) =$ 

$\ell(\gamma_n) = \ell_n$;

t_n = relative twist parameter at γ_n, $-\frac{1}{2} < t_n \le \frac{1}{2}$.

<u>Thm</u> ① $\sum e^{-(1-|t_n|)\frac{\ell_n}{2}} = \infty \Rightarrow X$ parabolic type

<u>Cor.</u>

(a) <u>$\frac{1}{2}$-twist</u> : $\sum e^{-\frac{\ell_n}{4}} = \infty \Rightarrow X$ parabolic type

(b) <u>Independent of twist</u> : $\sum e^{-\ell_n/2} = \infty \Rightarrow X$ parabolic type

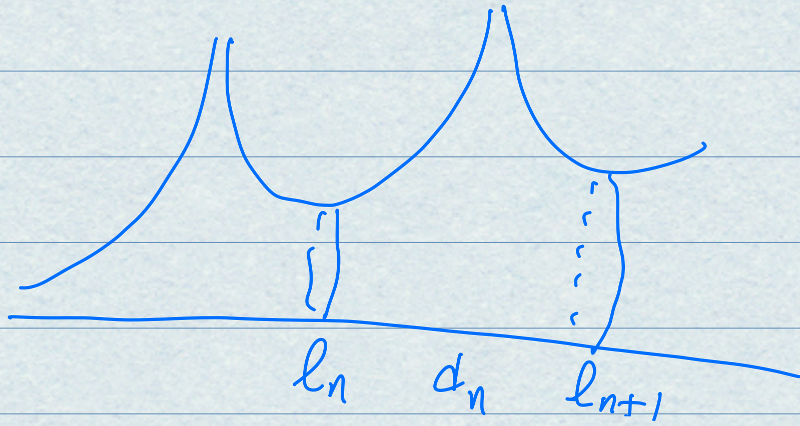
For zero twist flute we have a complete characterization:

Cor. (zero twists) Let $X = X(\{\ell_n, 0\})$. TFAE

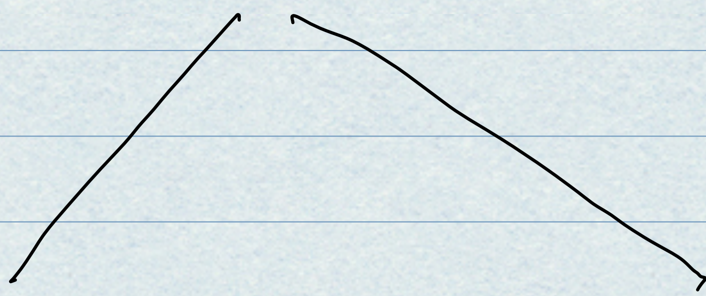
(1) X is parabolic type

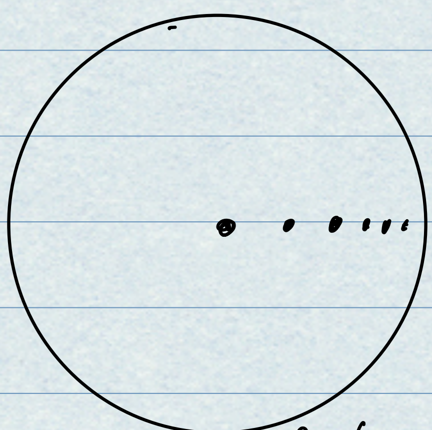
(2) $\sum e^{-\ell_n/2} = \infty$

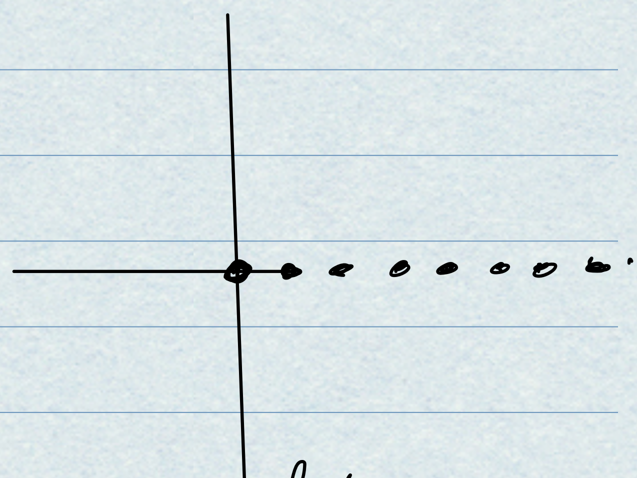
(3) X is geodesically complete



0-twist







$$\sum_n e^{-\ell_n/2} < \infty \qquad\qquad \sum_n e^{-\ell_n/2} = \infty$$

Tools of the Trade:

1) Extremal length of curve families

2) Collars (standard and nonstandard)

Extremal length: X a Riemann surface

Γ a family of curves on X.

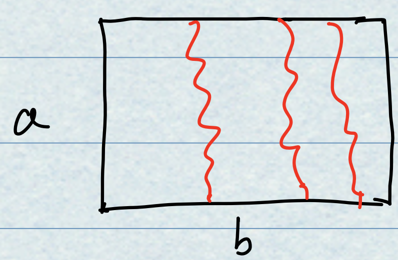
Extremal length of Γ,

$$\lambda(\Gamma) \equiv \sup_{\rho} \frac{L_\rho^2(\Gamma)}{A_\rho}, \quad \text{where}$$

$$L_\rho(\Gamma) = \inf_{\gamma \in \Gamma} \{\rho\text{-length of } \gamma\}$$

$$A_\rho = \rho\text{-Area of } X.$$

ex)



$$\lambda(\Gamma) = \frac{a}{b}$$

Properties of extremal length:

① Conformal invariant

[Smaller set, bigger E-length]

② If $\Gamma_1 \subset \Gamma_2$,

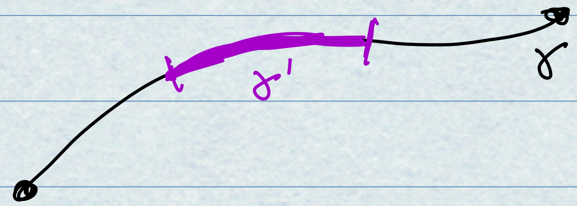
$\lambda(\Gamma_2) \leq \lambda(\Gamma_1)$

[overflowing]

③ Suppose Γ, Γ' curve families s.t.

$\forall \gamma \in \Gamma$, $\exists \gamma' \in \Gamma'$ where $\gamma' \subset \gamma$

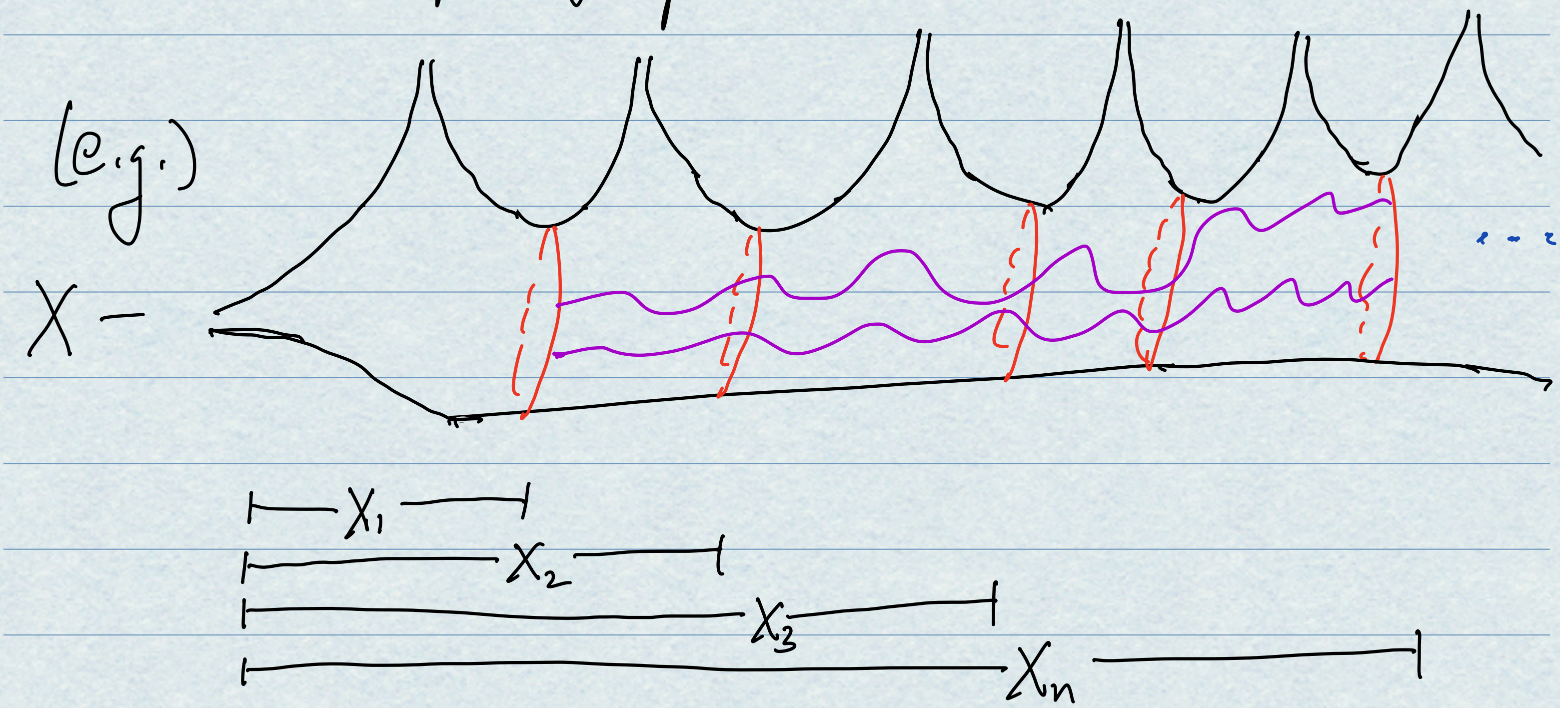
then $\lambda(\Gamma') \leq \lambda(\Gamma)$



Extremal length and type:

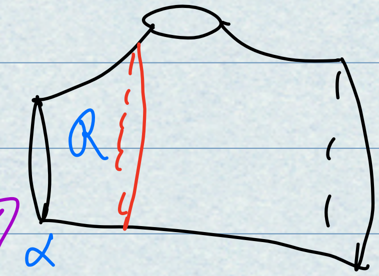
- $\{X_n\}$ a finite area exhaustion of X,
- Γ_n = family of curves from ∂X_1 to ∂X_n.

(e.g.)



[Ahlfors-Sario] (*) $\lambda(\Gamma_n) \to \infty \iff X$ of parabolic type

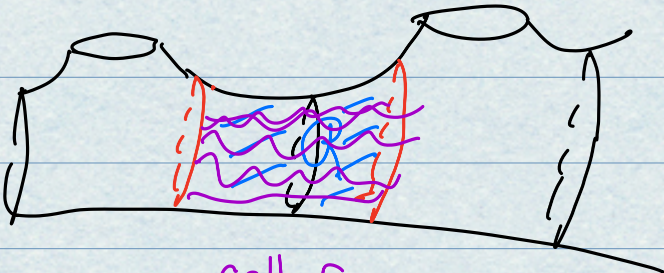
Collars:



R is topologically a cylinder containing α.

(one-sided collar)

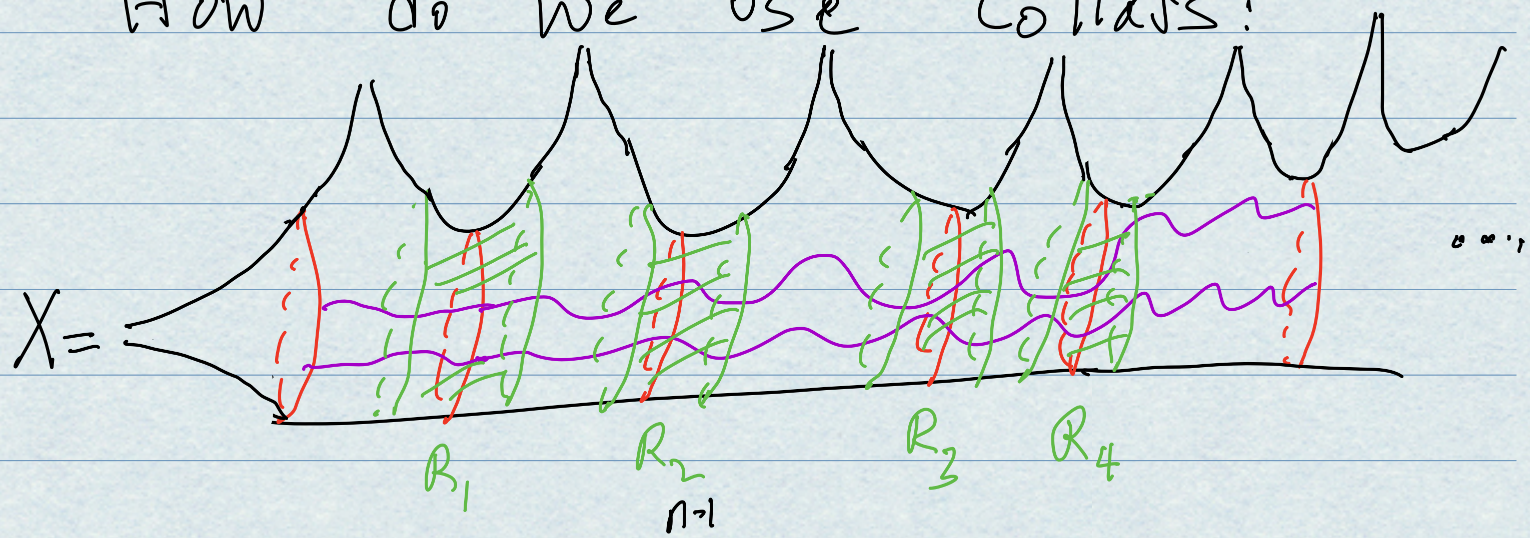
If α is a bdy. component of R, we say R is one-sided.



collar

Consider the curve family between boundary components of R, denoted $\lambda(R)$.

How do we use Collars:

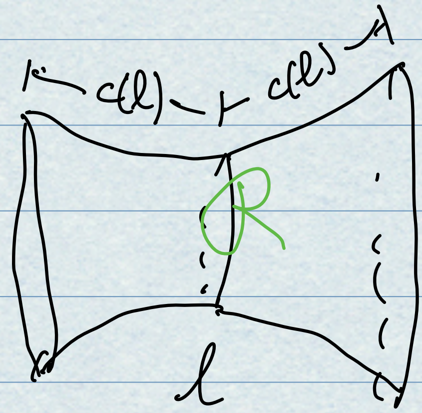


$$\lambda(\Gamma_n) \underset{\text{extremal length property}}{\geq} \sum_{k=1}^{n-1} \lambda(R_k)$$

So, $\sum_{k=1}^{\infty} \lambda(R_k) = \infty \Rightarrow X$ parabolic type

Goal: Find relationship between $\lambda(R)$ and F-N parameters.

standard Collar ;



standard collar width:

$$c(\ell) = \sinh^{-1}\left(\frac{1}{\sinh\frac{\ell}{2}}\right)$$

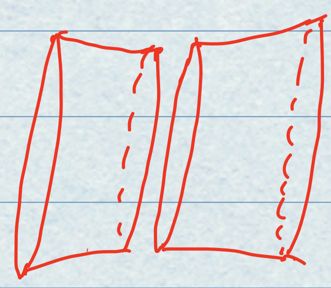
[Maskit] $\lambda(R) \geq c\,\frac{e^{-\ell/2}}{\ell}$

Remark:

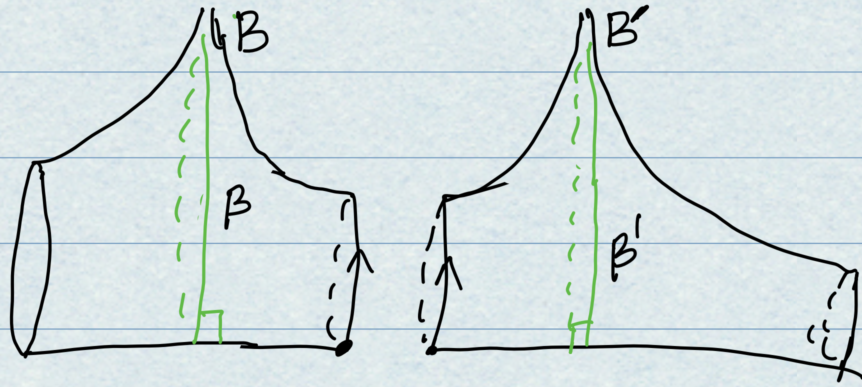
① Good estimate if $\ell \to 0$

② Twist does not matter:

that is,



nonstandard Collar



glue with twist t

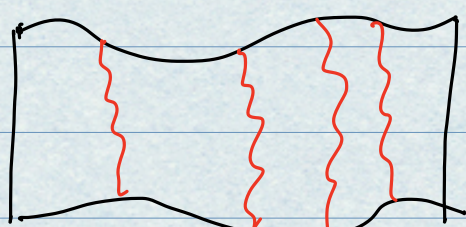
[β the geodesic from B to B
β' " " " B' to B']

with twist t

with mild hypothesis, we have

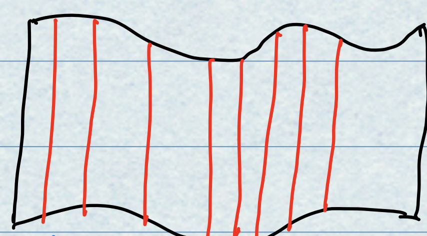
<u>Thm</u> (–, Hakobyan, Šarić)

$$\lambda(R) \geq c\, e^{-(1-|t(\alpha)|)\frac{\ell(\alpha)}{2}}$$



all curves top to bottom

coarsely approximate



vertical family

Thank You!!!